GKM-sheaves and nonorientable surface group representations

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Abstract

Let T be a compact torus and X a nice compact T-space (say a manifold or variety). We introduce a functor assigning to X a *GKM-sheaf* $\mathcal{F}_X$ over a *GKM-hypergraph* Γ_X. Under the condition that X is *equivariantly formal*, the ring of global sections of $\mathcal{F}_X$ are identified with the equivariant cohomology, $H_T^*(X;\mathbb{C}) \cong H^0(\mathcal{F}_X)$. We show that GKM-sheaves provide a general framework able to incorporate numerous constructions in the GKM-theory literature.

In the second half of the paper we apply these ideas to study the equivariant topology of the representation variety $\mathcal{R}_K := Hom(\pi_1(\Sigma), K)$ under conjugation by K, where Σ is a nonorientable surface and K is a compact connected Lie group. We prove that $\mathcal{R}_{SU(3)}$ is equivariantly formal for all Σ and compute its equivariant cohomology ring. We also show that $\mathcal{R}_{SU(5)}$ fails to be equivariantly formal in general, disproving a conjecture from [Bai08]

1 Introduction

1.1 GKM-sheaves

Let $T \cong U(1)^r$ be a compact torus of rank r. Equivariant cohomology[1] $H_T(-)$ is a functor from the category of T-spaces to the category of $\mathbb{Z}$-graded algebras over the cohomology ring of classifying space $H^*(BT)$. There is a canonical isomorphism

$$H^*(BT) \cong \mathbb{C}[\mathfrak{t}]$$

where $\mathfrak{t} = Lie(T) \otimes \mathbb{C}$ is the complexified Lie algebra viewed and $\mathbb{C}[\mathfrak{t}] \cong \mathbb{C}[x_1, ..., x_r]$ is the polynomial ring of regular functions, with grading $\deg(x_i) = 2$. A characteristic feature of equivariant cohomology theory is the interplay between the topology of a T-space X and the behavior of the $\mathbb{C}[\mathfrak{t}]$-module $H_T^*(X)$ under localization to linear subspaces of $\mathfrak{t}$.

A good example of this interplay is the **Borel localization theorem** (which comes in several versions). Suppose that X is a nice T-space, by which we mean a finite T-CW

[1] We use complex coefficients throughout.

complex[2] and let Y denote a nice T-subspace. Let $\mathfrak{t}_x$ denotes the complexified infinitesimal stabilizer of a point $x \in X$. Borel localization says that as a module over $\mathbb{C}[\mathfrak{t}]$, the support of $H_T^*(X,Y)$ satisfies

$$\mathrm{Supp}(H_T^*(X,Y)) \subseteq \bigcup_{x \in X \setminus Y} \mathfrak{t}_x. \tag{1}$$

Notice that by the finiteness assumption on X, the right hand side of (1) is equal to a finite union of linear spaces so it is an algebraic set.

An important special case is $Y = X^T$. Combining the localization theorem with the long exact sequence of the pair $i : X^T \hookrightarrow X$, we conclude that the kernel and cokernel of the homomorphism

$$i^* : H_T^*(X) \to H_T^*(X^T) \tag{2}$$

are torsion modules over $\mathbb{C}[\mathfrak{t}]$. In many interesting cases, $H_T^*(X)$ is known to be torsion free so $H_T^*(X) \cong \mathrm{im}(i^*)$ and it is useful to have effective methods of computing the image of i^*. One such method uses the Chang-Skjelbred Lemma [CS74] (see also [FP07]), which we now explain.

Let X be a nice T-space and consider the equivariant topological filtration

$$X^T = X_0 \subseteq X_1 \subseteq ... \subseteq X_r = X$$

where X_j is the union of T-orbits of dimension less than or equal to j. The pair (X_1, X_0) is called the **one-skeleton** of X. It follows from consideration of the long exact sequence of the pair that the image of (2) lands in the kernel of the boundary map

$$d : H_T^*(X_0) \to H_T^*(X_1, X_0) \tag{3}$$

so we may restrict i^* to a homomorphism

$$\bar{i^*} : H_T^*(X) \to \ker(d). \tag{4}$$

If $H_T^*(X)$ is torsion free, then it is not hard to show (see Theorem 2.6) that $\bar{i^*}$ is injective, with cokernel supported in codimension at least two. The **Chang-Skjelbred Lemma** states that if $H_T^*(X)$ is a *free* $\mathbb{C}[\mathfrak{t}]$-module, then $\bar{i^*}$ is an isomorphism.

This hypothesis is sufficiently important to warrant it own name. A T-space X is called **equivariantly formal** if $H_T^*(X)$ is a free $\mathbb{C}[\mathfrak{t}]$-module. In this case there is an isomorphism of $\mathbb{Z}$-graded $\mathbb{C}[\mathfrak{t}]$-modules,

$$H_T^*(X) \cong H^*(X) \otimes \mathbb{C}[\mathfrak{t}].$$

Many interesting examples of T-spaces are known to be equivariantly formal, including compact Hamiltonian T-manifolds and spaces whose ordinary cohomology lies only in even degrees.

[2] This class of T-spaces include compact, smooth T-manifolds and compact, real algebraic subsets of linear T representations [PS98].

GKM theory translates the problem of computing the image of (3) into combinatorial algebra. Suppose that X is a nonsingular projective T-equivariant variety for which X_0 is a finite set of points and X_1 is a finite union of rational curves linking pairs of points in X_0. The combinatorics the 1-skeleton (X_1, X_0) is neatly described using a graph Γ_X, called the *GKM-graph* of X. The vertices of Γ_X correspond to points of X_0 and the edges of Γ_X correspond to the irreducible components of X_1. The edges are labelled by the character $\alpha : T \to U(1)$ through which T acts on the corresponding rational curve (this character is only determined up to sign). GKM theory provides a recipe for computing $H_T^*(X) = \ker(d_1^0)$ from this labelled graph, first described in [GKM97] (see also [GZ01]).

One of the main results of the current paper is to generalize this GKM procedure to all nice T-spaces X. In §2.1 we define a functor $X \mapsto \Gamma_X$ from the category of nice T-spaces to the category of *GKM-hypergraphs*, so that Γ_X encodes the combinatorics of the one-skeleton (X_1, X_0). In §2.2 we define the notion of a *GKM-sheaf* over a GKM-hypergraph, and associate to any nice T-space X a GKM-sheaf $\mathcal{F}_X$ over Γ_X such that the algebra of global sections $H^0(\mathcal{F}_X)$ is naturally isomorphic to $\ker(d)$ from (4). If $H_T^*(X)$ is free, then

$$H^0(\mathcal{F}_X) \cong H_T^*(X) \tag{5}$$

by the Chang-Skjelbred Lemma.

GKM-sheaves are related to and were inspired by the Γ-sheaves of Braden-MacPherson [BM01], which used in the study of equivariant intersection cohomology. In §2.5 we show that every *pure* Γ-sheaf determines a GKM-sheaf in such a way that their modules of global sections are isomorphic. GKM-sheaves are also a useful framework for better understanding constructions due to Guillemin-Zara [GZ03] and Guillemin-Holm [GH04], related to the topology of Hamiltonian actions, a point we address in §2.5.

1.2 Representation varieties

Let $\pi = \pi_1(\Sigma)$ be the fundamental group of a compact, nonorientable 2-manifold without boundary and let K be a compact, connected Lie group of rank r. The space of homomorphisms $Hom(\pi, K)$ equipped with the compact open topology is called a *representation variety*. The group K acts by conjugation on $Hom(\pi, K)$ producing the topological quotient stack, $[Hom(\pi, K)/K]$ which is naturally isomorphic to the moduli stack of flat K-bundles over Σ.

An important invariant of a stack is its stack cohomology, which in our case can be identified with the equivariant cohomology $H_K^*(Hom(\pi, K))$, and this is the object of study in the second half of this paper (In fact, we work with a wider class of varieties associated to a punctured surface that includes to $Hom(\pi, K)$ as a special case).

In [Bai10], the author derived formulas for the cohomology rings

$$H^*([Hom(\pi, SU(2))/SU(2)]) = H^*_{SU(2)}(Hom(\pi, SU(2))),$$

describing Poincaré polynomials and cup product structure (unless otherwise indicated, cohomology is singular with complex coefficients). This calculation was greatly facilitated by the following fact:

Theorem 1.1. *For $K = SU(2)$, the K-space $Hom(\pi, K)$ is equivariantly formal.*

Recall that a nice K-space X is called equivariantly formal if $H^*_K(X)$ is free as a module over $H^*(BK)$ or equivalently, if $H^*(X) \otimes_{\mathbb{C}} H^*(BK)$ as graded vector spaces. The second objective of the current paper is to test whether this equivariant formality property generalizes other to Lie groups. The answer is no in general: it fails when $K = SU(5)$ and Σ is the Klein bottle. However we do get a positive result for $K = SU(3)$.

Theorem 1.2. *The $SU(3)$-space $Hom(\pi, SU(3))$ is equivariantly formal, with equivariant Poincaré series*

$$\frac{(1+t^3+t^5+t^8)^g + (1+t^2+t^4)(t^3+2t^4+t^5)^g}{(1-t^2)(1-t^4)(1-t^6)}. \tag{6}$$

We also obtain an explicit description of the cup product structure on $H^*_{SU(3)}(Hom(\pi, SU(3)))$.

The formula (6) for the Poincaré series was conjectured by Ho and Liu [HL08] and later proven by the author [Bai09] using the Morse theory of the Yang-Mills functional, but this approach concealed the cup product structure. On the other hand, GKM-theory respects the cup product structure so by combining GKM methods with the previously obtained formula (6) we are able to prove Theorem 1.2. We predict that further application of Yang-Mills theory will establish the following.

Conjecture 1. Equivariant formality holds for compact, connected Lie groups K whose simple summands have rank ≤ 2. This includes in particular types B_2 and G_2.

Now we outline the strategy. Recall that if $T \subset K$ is a maximal torus and $W = N(T)/T$ the Weyl group, then for any T-space X, there is a canonical isomorphism

$$H_K(X) = H^*_T(X)^W,$$

so to a great extent the study of connect compact groups actions reduces to torus actions. Because T is maximal abelian in K it follows easily that

$$X_0 := Hom(\pi, K)^T = Hom(\pi, T).$$

The codimension one tori in T that occur as stabilizer groups of the torus action are precisely the root hypertori of T in K. Thus if we let Δ_+ denote the set of positive roots of K, we have

$$X_1 = \bigcup_{\alpha \in \Delta_+} Hom(\pi, K_\alpha) \tag{7}$$

where K_α is the centralizer of the kernel of α. The groups K_α have semisimple rank 1, so admit a covering $U(1)^{r-1} \times SU(2) \to K_\alpha$, where r is the rank of K. The representation variety $Hom(\pi, K_\alpha)$ can then be described using the known properties of $Hom(\pi, SU(2))$ from [Bai10]. Since we have a good understanding of the one-skeleton (X_1, X_0), we are able to compute the global sections of the GKM sheaf $\mathcal{F}_X$. If $H^*_T(X)$ is equivariantly formal, then necessarily $H^0(\mathcal{F}_X)$ must be free, so computing $H^0(\mathcal{F}_X)$ provides a test of the equivariant formality of $H^*_T(X)$. In the case of $K = SU(3)$, this is combined with Morse theoretic information to compute $H^*_T(X)$.

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2 GKM theory

Fix a compact torus T with complexified Lie algebra $\mathfrak{t} = Lie(T) \otimes \mathbb{C}$ and dual $\mathfrak{t}^* = Hom(\mathfrak{t}, \mathbb{C})$. Let $A := \mathbb{C}[\mathfrak{t}] \cong S(\mathfrak{t}^*)$ denote the symmetric algebra of polynomial functions on $\mathfrak{t}$, graded so that $\mathfrak{t}^*$ has degree 2. Then as a graded ring, A is naturally isomorphic to $H^*(BT)$. We say that a T-space X is *nice* if it admits the structure of a finite $T-CW$ complex.

2.1 GKM-hypergraphs

A **hypergraph**[3] $\Gamma = (\mathcal{V}, \mathcal{E}, I)$, consists of a pair of sets $\mathcal{V}$ and $\mathcal{E}$ called the **vertices** and **hyperedges** of Γ respectively, and a map

$$I : \mathcal{E} \to \mathcal{P}(\mathcal{V}) \setminus \emptyset$$

called the **incidence map**, where $\mathcal{P}(\mathcal{V})$ is the power set of $\mathcal{V}$. We say that $v \in \mathcal{V}$ is **incident** to $e \in \mathcal{E}$ if $v \in I(e)$. A **graph** is a hypergraph for which no hyperedge is incident to more than two vertices.

Let

$$\Lambda := Hom(T, U(1))$$

be the weight lattice of T, which we think of as embedded in $\mathfrak{t}^*$ in the usual way. Let $\mathbb{P}(\Lambda)$ denote set of **projective weights**, the set of non-zero weights modulo scalar multiplication. The elements of $\mathbb{P}(\Lambda)$ are in one-to-one correspondence with the codimension one subtori of T by the rule

$$\alpha_0 \in \mathbb{P}(\Lambda) \leftrightarrow \ker(\tilde{\alpha}_0) \subset T$$

where $\tilde{\alpha}_0 \in \Lambda$ is a primitive representative of α_0. We will generally be sloppy and write α_0 in place of $\tilde{\alpha}_0$.

Definition 1. Let $(\mathcal{V}, \mathcal{E}, I)$ be a hypergraph with finite $\mathcal{V}$. An **axial function** is a surjective map

$$\alpha : \mathcal{E} \to \mathbb{P}(\Lambda), \quad e \to \alpha(e)$$

such that for each $\alpha_0 \in \mathbb{P}(\Lambda)$ the sets $\{I(e) | \alpha(e) = \alpha_0\}$ form a partition of $\mathcal{V}$. We call $(\mathcal{V}, \mathcal{E}, I, \alpha)$ a **GKM-hypergraph**.

Remark 1. The data defining a GKM-hypergraph is equivalent to

- A finite set of vertices $\mathcal{V}$
- A partition of $\mathcal{V}$ for each projective weight in $\alpha_0 \in \mathbb{P}(\Lambda)$.

and we will sometimes construct GKM-hypergraphs by describing this equivalent data.

[3] Note that we do not insist that the incidence map is injective, which is often required of hypergraphs.

Denote by $\mathcal{E}^{\alpha_0} = \alpha^{-1}(\alpha_0)$ the inverse image of $\alpha_0 \in \mathbb{P}(\Lambda)$, so

$$\mathcal{E} = \coprod_{\alpha_0 \in P(\Lambda)} \mathcal{E}^{\alpha_0}. \tag{8}$$

Remark 2. It is clear from (8) that GKM-hypergraphs always have uncountably many hyperedges if the rank of T is greater than one. This differs from other definitions of GKM-graphs in the literature, which normally require the edge set to be finite. Our definition is not really a substantial departure from the literature in this respect, because in practice all but a finite number of hyperedges will be **degenerate**, meaning incident only to a single vertex.

As indicated in the introduction, GKM-hypergraphs are designed to capture combinatorial information about the one-skeletons of T-spaces.

Definition 2. For any nice T-space X we associate a GKM-hypergraph Γ_X using the equivalent data of Remark 1. The vertex set

$$\mathcal{V} \cong \pi_0(X^T)$$

equals the set of connected components of X^T, and for each $\alpha_0 \in \mathbb{P}(\Lambda)$ the partition of $\mathcal{V}$ is determined by the equivalence relation

$$V_1 \sim_{\alpha_0} V_2$$ [4]

if and only if V_1 and V_2 lie in the same connected component of $X^{\ker(\alpha_0)}$. The elements of $\mathcal{E}^{\alpha_0}$ are in one-to-one correspondence with the connected components of $X^{\ker(\alpha_0)}$ that contain fixed points and we will represent the vertices and spaces by the corresponding subspaces of X.

A couple of examples of nice T-spaces to keep in mind are:

Example 1. Let X be a non-singular, projective toric variety. Then X^T is a finite set, $X^{\ker(\alpha_0)}$ is a finite union of points and rational curves for each $\alpha_0 \in \mathbb{P}(\Lambda)$, and T acts on each rational curve by rotation. The hypergraph Γ_X a graph, with vertices $\mathcal{V} = X^T$ and with (nondegenerate) edges corresponding to the rational curves.

Example 2. A compact Lie group K acted on via conjugation by its maximal torus T. There is only one vertex, so all edges are degenerate. We call this *the* GKM-hypergraph with one vertex.

Definition 3. A **morphism of** GKM**-hypergraphs**, $\phi : (\mathcal{V}, \mathcal{E}, I, \alpha) \to (\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{I}, \tilde{\alpha})$, is a map of sets

$$\phi : \mathcal{V} \cup \mathcal{E} \to \tilde{\mathcal{V}} \cup \tilde{\mathcal{E}}$$

sending $\mathcal{V}$ to $\tilde{\mathcal{V}}$ and $\mathcal{E}$ to $\tilde{\mathcal{E}}$ and satisfying for every $e \in \mathcal{E}$

$$\alpha(e) = \tilde{\alpha}(\phi(e)) \quad and \quad \phi(I(e)) \subseteq \tilde{I}(\phi(e)).$$

[4] We will tend to denote vertices and hyperedges in upper case V, E when they are represented by spaces, and lower case v, e for abstract hypergraphs.

Proposition 2.1. *The rule $X \mapsto \Gamma_X$ extends to a functor from nice T-spaces to GKM-hypergraphs.*

Proof. If $\Phi : X \to Y$ is a T-equivariant map of nice T-spaces, then the functorial morphism $\phi : \Gamma_X \to \Gamma_Y$ is induced as follows.

The map Φ restricts to a map $X^T \to Y^T$, inducing the map

$$\phi|_{\mathcal{V}} : \mathcal{V}_X \to \mathcal{V}_Y$$

on connected components. Similarly Φ restricts to maps $X^{\ker(\alpha_0)} \to Y^{\ker(\alpha_0)}$ for each α_0, inducing

$$\phi|_{\mathcal{E}^{\alpha_0}} : \Gamma_X \to \Gamma_Y$$

on connected components. The condition $\phi(I_X(e)) \subseteq I_Y(\phi(e))$ is a consequence of the fact that if $x_1, x_2 \in X^T$ lie in the same connected component of $X^{\ker(\alpha_0)}$, then $\Phi(x_1)$ and $\Phi(x_2)$ lie in the same component of $Y^{\ker(\alpha_0)}$. $\square$

Given a hypergraph $(\mathcal{V}, \mathcal{E}, I)$, we define a topology on the union $\mathcal{V} \cup \mathcal{E}$, by the rule that $S \subseteq \mathcal{V} \cup \mathcal{E}$ is open if for every $e \in S \cap \mathcal{E}$, we have $I(e) \subset S$. This topology is generated by **basic open sets**

$$U_v := \{v\}$$

for $v \in \mathcal{V}$ and

$$U_e := \{e\} \cup I(e)$$

for $e \in \mathcal{E}$. Observe that, counterintuitively, vertices are open points and hyperedges are closed points.

GKM-hypergraph morphisms possess the following useful property.

Proposition 2.2. *Let $\phi : \Gamma \to \tilde{\Gamma}$ be a morphism of GKM-hypergraphs. The preimage of any basic open set $U_x \subset \tilde{\Gamma}$ is a disconnected union of basic open sets:*

$$\phi^{-1}(U_x) = \coprod_{y \in \phi^{-1}(x)} U_y$$

In particular, $\phi : \Gamma \to \tilde{\Gamma}$ is a continuous map.

Proof. This is clear for the singleton sets $U_v = \{v\}$ centered on vertices. For a hyperedge $\tilde{e} \in \tilde{\Gamma}$, we have $U_{\tilde{e}} = \tilde{e} \cup \tilde{I}(\tilde{e})$. So we only need to show if $\phi(v) \in \tilde{I}(\tilde{e})$ then there exists $e \in \Gamma$ such that $v \in I(e)$ and $\phi(e) = \tilde{e}$.

By the partition condition on GKM-hypergraphs, there is a unique $e \in \mathcal{E}$ such that $\alpha(e) = \tilde{\alpha}(\tilde{e})$ and $v \in I(e)$. Thus $\tilde{v} = \phi(v) \in \phi(I(e)) \subseteq \tilde{I}(\phi(e))$ and since $\tilde{v} \in \tilde{I}(\tilde{e})$ we deduce that $\phi(e) = \tilde{e}$.

$\square$

2.2 GKM-sheaves

We begin with an abstract definition.

Definition 4. Let $\mathcal{F}$ be a sheaf of finitely generated, $\mathbb{Z}$-graded A-modules over the underlying topological space of a GKM-hypergraph, $\Gamma = (\mathcal{V}, \mathcal{E}, I, \alpha)$. Then $\mathcal{F}$ is called a **GKM-sheaf** if the following conditions hold.

- $\mathcal{F}$ is locally free over A (that is, for every $x \in \Gamma$ the stalk $\mathcal{F}(U_x)$ is a free A-module).
- For all $e \in \mathcal{E}$, the restriction map $res_e : \mathcal{F}(U_e) \to \mathcal{F}(I(e))$ becomes an isomorphism upon inverting $\alpha(e)$:

$$\mathcal{F}(U_e) \otimes_A A[\alpha(e)^{-1}] \cong \mathcal{F}(I(e)) \otimes_A A[\alpha(e)^{-1}] \tag{9}$$

  .
- $res_e : \mathcal{F}(U_e) \to \mathcal{F}(I(e))$ is an isomorphism for all but a finite number of $e \in \mathcal{E}$.

We denote by $GKM(\Gamma)$ the full subcategory of the category of sheaves of graded A-modules on Γ, whose objects are GKM-sheaves.

Any nice T-space X determines a sheaf of graded A-algebras on Γ_X, denoted $\mathcal{F}_X$, with stalks

$$\mathcal{F}_X(U_V) = \mathcal{F}_X(V) = H_T^*(V)$$

at vertices and

$$\mathcal{F}_X(U_E) = \mathcal{F}_X(E \cup I(E)) = H_T^*(E)/Tor_A(H_T^*(E))$$

at edges, where

$$Tor_A(M) := \{m \in M |\ am = 0 \text{ for some } a \in A \setminus \{0\}\}$$

denotes the torsion submodule of the A-module M. If $i : V \subset E$ is a subset then the restriction map $res_E : \mathcal{F}_X(U_E) \to \mathcal{F}_X(U_V)$ induced by cohomology morphism $i^* : H_T^*(E) \to H_T^*(V)$. This is well defined because $H_T^*(V)$ is torsion-free so $Tor_A(H_T^*(E)) \subseteq \ker(i^*)$.

Proposition 2.3. *If X is a nice T-space then $\mathcal{F}_X$ is a GKM sheaf.*

Proof. For any $E \in \mathcal{E}$, the restriction map $res_E : \mathcal{F}_X(U_E) \to \mathcal{F}(I(E))$ is identified with the map $H_T^*(E)/Tor_A(H_T^*(E)) \to H_T^*(E^T)$ induced by the inclusion $E^T \subset E$. Since the only isotropy algebra for $E \setminus E^T$ is $(\alpha_E)^\perp \subset \mathfrak{t}$, the Borel localization theorem (1) tells us that the kernel and cokernel of res_E are α_E-torsion.

The finiteness condition follows easily from compactness of X. Local freeness of $\mathcal{F}_X$ is an immediate consequence of Lemma 2.4. $\square$

Lemma 2.4. *If X is a nice T-space and $H \subset T$ is a codimension one subtorus, then $H_T^*(X^H)$ is the direct sum of a free and a torsion A-module. If moreover, $H_T^*(X)$ is torsion-free, then $H_T^*(X^H)$ is free.*

Proof. Let $\alpha_0 \in \Lambda \subset \mathfrak{t}^*$ be the character for which H is the kernel. Because H acts trivially on X^H, $H_T^*(X^H) \cong H_{T/H}^*(X^H) \otimes_{\mathbb{C}[\alpha_0]} A$. Since $\mathbb{C}[\alpha_0]$ is a PID, the fundamental theorem of finitely generated modules over a PID implies that $H_{T/H}^*(X^H)$ (hence also $H_T^*(X^H)$) is isomorphic to the sum of a free module and a torsion module.

Furthermore, because $H_{T/H}^*(X^H)$ is graded $\mathbb{C}[\alpha_0]$-module, its torsion modules decomposes into a direct sum of modules of the form

$$\mathbb{C}[\alpha_0]/\alpha_0^n\mathbb{C}[\alpha_0]$$

for some positive integer n (up to degree shifts). Thus $H_T^*(X^H)$ is free if and only if it does not contain a summand isomorphic to $A/\alpha_0^n A$.

Now assume that $H_T^*(X)$ is torsion free over A. Localizing at the prime ideal $(\alpha_0) \subset A$ we have

$$H_T^*(X)_{(\alpha_0)} \cong H_T^*(X^H)_{(\alpha_0)}$$

by Borel localization. Localization is an exact functor so it preserves torsion-freeness and we infer that $H_T^*(X^H)_{(\alpha_0)}$ is torsion free over $A_{(\alpha_0)}$. It follows that $H_T^*(X^H)$ can not contain a summand of the form $A/\alpha_0^n A$, so we conclude that $H_T^*(X^H)$ is free over A. $\square$

Notice that $\mathcal{F}_X$ is in fact a sheaf of A-algebras, so its set of global sections $H^0(\mathcal{F}_X)$ is an A-algebra.

Recall from §1.1 the one-skeleton (X_1, X_0) of a T-space X.

Lemma 2.5. *Let X be a nice T-space. The space of global sections $H^0(\mathcal{F}_X)$ fits into an exact sequence of A-modules*

$$0 \to H^0(\mathcal{F}_X) \xrightarrow{r} H_T^*(X_0) \xrightarrow{d} H_T^*(X_1, X_0)[+1].$$

for which r is a homomomorphism of A-algebras.

Proof. The map r is identified with the sheaf restriction map $H^0(\mathcal{F}_X) \to \mathcal{F}_X(\mathcal{V}) \cong H_T^*(X_0)$, so r is certainly a homomorphism of algebras.

Since $\mathcal{F}_X$ is locally free and the restriction maps $res_E : \mathcal{F}_X(U_E) \to \mathcal{F}_X(I(E))$ are isomorphisms modulo torsion, they must be injective. Thus any element of $\mathcal{F}_X(\mathcal{V})$ extends in at most one way to each hyperedge and we deduce that r is injective.

It remains to show that $\mathrm{im}(r) = \ker(d)$. Decomposing cohomology into connected components we have an isomorphism

$$H_T^*(X_0) \cong \bigoplus_{V \in \mathcal{V}} H_T^*(V)$$

and

$$H_T^*(X_1, X_0) \cong \Big(\bigoplus_{E \in \mathcal{E}} H_T^*(E, E^T)\Big) \oplus H_T^*(X_1')$$

where X_1' is the union of components of X_1 that do not intersect X_0. Clearly the projection of d onto $H_T^*(X_1')$ is zero, so $\ker(d) \cong \ker(d')$ where d' is the block decomposition

$$d' : \bigoplus_{V \in \mathcal{V}} H_T^*(V) \to \bigoplus_{E \in \mathcal{E}} H_T^{*+1}(E, E^T). \tag{10}$$

with matrix blocks $d'_{E,V}$, such that $d'_{E,V} = 0$ when $V \not\subseteq E$, and when $V \subseteq E$ the diagram

$$\begin{tikzcd}
H_T^*(V) \arrow[rr, "d'_{E,V}"] \arrow[dr, "\iota"'] & & H_T^{*+1}(E, E^T) \\
& H_T^*(E^T) \arrow[ur, "\delta"'] &
\end{tikzcd} \tag{11}$$

commutes, where δ is the boundary map of the pair (E, E^T) and ι is inclusion as a summand.

Thus $(a_V) \in \bigoplus_{V \in \mathcal{V}} H_T^*(V)$ lies in $\ker(d')$ if and only if for all $E \in \mathcal{E}$,

$$0 = \sum_{V \in \mathcal{V}} d'_{E,V}(a_V) = \sum_{V \subseteq E} d'_{E,V}(a_V) = \delta(\sum_{V \subseteq E} \iota(a_V)). \tag{12}$$

By the long exact sequence of the pair (E, E^T), (12) holds if and only if $\sum_{V \subseteq E^T} \iota(a_V) \in H_T^*(E^T)$ lies in the image of $H_T^*(E) \to H_T^*(E^T)$ and this is equal to the image of $res_E : \mathcal{F}_X(U_E) \to \mathcal{F}(I(E))$. Thus $\ker(d')$ corresponds to those sections $\mathcal{F}(\mathcal{V})$ that can extend to every hyperedge and we conclude that $\mathrm{im}(r) = \ker(d') = \ker(d)$.

□

Recall the filtration $X_0 \subseteq X_1 \subseteq ... \subseteq X_r = X$ described in §1.1. This filtration is preserved by T, so it determines a spectral sequence $(E_r^{p,q}, d_r)$ converging to $H_T^*(X)$ (see [FP07]). The first page of the spectral sequence satisfies

$$E_1^{p,q} \cong H_T^{p+q}(X_p, X_{p-1})$$

and $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ equals the coboundary operator of the triple (X_{p+1}, X_p, X_{p-1}). Thus $H^0(\mathcal{F}_X)$ computes the column $E_2^{0,*}$.

Theorem 2.6. *Let X be a nice T-space. There is a natural short exact sequence of graded A-algebras,*

$$0 \to Tor_A(H_T^*(X)) \to H_T^*(X) \xrightarrow{\psi} H^0(\mathcal{F}_X) \to Q(X),$$

where $Q(X)$ is an A-module with support in codimension ≥ 2. If X is equivariantly formal, then ψ is an isomorphism

$$H_T^*(X) \overset{\psi}{\cong} H^0(\mathcal{F}_X).$$

Proof. First, note that $H^0(\mathcal{F}_X)$ is torsion free, so $Tor_A(H_T^*(X)) \subseteq \ker(\psi)$. Next, observe that because the spectral sequence $E_r^{p,q}$ converges to $H_T^*(X)$, then $\ker(\psi)$ admits a finite filtration whose associated graded object is isomorphic to a subquotient of the A-module $\oplus_{p=1}^r H_T^*(X_p, X_{p-1})$, which is a torsion module by the localization theorem (1). We deduce that $\ker(\psi)$ is torsion and that $\ker(\psi) = Tor_A(H_T^*(X))$.

Similarly, an associated graded version of $\mathrm{cok}(\psi)$ is equal to

$$\bigoplus_{p=2}^{r} \mathrm{im}(d_p : E_p^{0,*} \to E_p^{p,(*-p+1)})$$

which is a subquotient of $\bigoplus_{p=2}^{r} H_T^*(X_p, X_{p-1})$, which by the localization theorem has support in codimension two.

If X is equivariantly formal, then the Chang-Skjelbred Lemma says that ψ is an isomorphism. $\square$

Both of the following example are equivariantly formal, so Theorem 2.6 holds for them.

Example 3. The GKM-sheaf of a toric variety has stalks $\mathcal{F}_X(U_x) = H_T^*(pt) = A$ when x is a vertex or degenerate edge, and $\mathcal{F}_X(U_e) \cong H_T^*(S^2) \cong A \oplus A[2]$ as and A-module, when e is a nondegenerate edge. We use the grading shift convention $M[d]^* \cong M^{*-d}$. The restriction map at a nondegenerate edge is the map from $\mathcal{F}_X(U_e) \cong A \oplus A[2]$ to $\mathcal{F}_X(I(e)) \cong A \oplus A$ defined by the matrix

$$\begin{pmatrix} 1 & \alpha(e) \\ 1 & -\alpha(e) \end{pmatrix}.$$

Example 4. The GKM-sheaf of a compact, connected Lie group K with T a maximal torus has one vertex with stalk $\mathcal{F}_K(U_v) = H_T^*(T) \cong \bigwedge \mathfrak{t}^* \otimes_{\mathbb{C}} A$. For $\alpha(e)$ a root of K, the stalk equals

$$\mathcal{F}_{K,e} = H_T^*(K^H) = \Big((\wedge \mathfrak{h}^*) \otimes A\Big) \oplus \Big(\alpha(e)(\wedge \mathfrak{h}^*) \oplus \alpha(e)A\Big),$$

where $H = \ker(\alpha(e))$ and $\mathfrak{h} = Lie(H) \otimes \mathbb{C}$. For all other values of $\alpha(e)$, $\mathcal{F}_K(U_e) = \mathcal{F}_K(U_v) = \wedge \mathfrak{t}^* \otimes A$. The restriction map is the obvious inclusion for all edges.

There is another description of the global sections that will prove useful. For each $\alpha_0 \in \mathbb{P}(\Lambda)$, denote be $i^*_{\alpha_0}$ the restriction map from $\mathcal{F}(\mathcal{V} \cup \mathcal{E}^{\alpha_0})$ to $\mathcal{F}(\mathcal{V})$. Note that by the finiteness condition in Definition 4, $i^*_{\alpha_0}$ is an isomorphism for all but a finite set of $\alpha_0 \in \mathbb{P}(\Lambda)$.

Lemma 2.7. *The restriction map $i^* : H^0(\mathcal{F}) \to \mathcal{F}(\mathcal{V})$ is injective, with image*

$$\mathrm{im}(i^*) = \bigcap_{\alpha_0 \in \mathbb{P}(\Lambda)} \mathrm{im}(i^*_{\alpha_0}). \tag{13}$$

Proof. Because $\mathcal{F}$ is locally free, $\mathcal{F}(\mathcal{V})$ is a free A-module and $H^0(\mathcal{F})$ is a torsion-free A-module. This implies that i^* is injective or equivalently, that any section $\mathcal{F}(\mathcal{V})$ extends in at most one way to $\mathcal{F}(\Gamma) = H^0(\mathcal{F})$. Both sides of (13) equal the set of sections in $\mathcal{F}(\mathcal{V})$ that extend to global sections. $\square$

2.3 Operations on GKM sheaves

2.3.1 Pushforwards

Recall that for a continuous map $f : X \to Y$ and a sheaf $\mathcal{F}$ on X, that the **pushforward sheaf** $f_*(\mathcal{F})$ is defined by the rule $f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ for open sets $U \subseteq Y$.

Proposition 2.8. *Let $f : \Gamma \to \tilde{\Gamma}$ be a morphism of GKM-hypergraphs and $\mathcal{F}$ a GKM-sheaf over Γ. The push-forward sheaf, $f_*(\mathcal{F})$, is a GKM-sheaf over $\tilde{\Gamma}$ satisfying*

$$H^0(\mathcal{F}) = H^0(f_*(\mathcal{F})). \tag{14}$$

Proof. For any $y \in \tilde{\Gamma}$ we have by Proposition 2.2 that

$$f_*(\mathcal{F})(U_y) = \bigoplus_{x \in f^{-1}(y)} \mathcal{F}(U_x)$$

which is a direct sum of free modules, hence free. By the same proposition, for a hyperedge $\tilde{e} \in \tilde{\Gamma}$, the restriction map $res_{\tilde{e}} : f_*(\mathcal{F})(U_{\tilde{e}}) \to f_*(\mathcal{F})(\tilde{I}(\tilde{e}))$ is identified with the direct sum of maps

$$\bigoplus_{e \in f^{-1}(\tilde{e})} \Big(res_e : \mathcal{F}(U_e) \to \mathcal{F}(I(e)) \Big)$$

which is an isomorphism modulo $\alpha(e)$ and is an isomorphism for all but finitely many $\tilde{e} \in \tilde{\Gamma}$.

Equation (14) holds simply by the definition of the pushforward sheaf. □

Proposition 2.9. *For any GKM-hypergraph morphism $\phi : \Gamma_1 \to \Gamma_2$ the pushforward ϕ_* of sheaves defines a functor $\phi_* : GKM(\Gamma_1) \mapsto GKM(\Gamma_2)$.*

Proof. This follows from the fact that push-forward is a functor for sheaves of A-modules. □

The assignment $X \mapsto \mathcal{F}_X$ is functorial in the following sense.

Proposition 2.10. *Let $\Phi : X \to Y$ be a T-equivariant map between nice T-spaces and let $\phi : \Gamma_X \to \Gamma_Y$ be the induced morphism of GKM-hypergraphs. There is a natural morphism $h : \mathcal{F}_Y \to \phi_*(\mathcal{F}_X)$ for which the following diagram commutes*

$$\begin{tikzcd} H_T^*(Y) \arrow[r, "\phi^*"] \arrow[d] & H_T^*(X) \arrow[d] \\ H^0(\mathcal{F}_Y) \arrow[r, "H^0(h)"] & H^0(\phi_*(\mathcal{F}_X)) = H^0(\mathcal{F}_X) \end{tikzcd} \tag{15}$$

where the vertical arrows come from Theorem 2.6.

Proof. A T-equivariant map $\Phi : X \to Y$ restricts to a map of the pairs $(X_1, X_0) \to (Y_1, Y_0)$. By Lemma 2.5, there is an induced map $l : H^0(\mathcal{F}_Y) \to H^0(\mathcal{F}_X)$ which fits into the commutative diagram in place of $H^0(h)$. It remains to find a GKM-sheaf morphism $h : \mathcal{F}_Y \to \phi_*(\mathcal{F}_X)$ such that $H^0(h) = l$.

If $\tilde{V}$ is a connected component of $Y_0 = Y^T$ representing a vertex of Γ_Y, then at the level of stalks we have

$$\mathcal{F}_Y(U_{\tilde{V}}) = H_T^*(\tilde{V}) \to \bigoplus_{V \in \phi^{-1}(\tilde{V})} H_T^*(V) = \phi_*(\mathcal{F}_X)(U_{\tilde{V}})$$

is the direct sum of maps $H_T^*(\tilde{V}) \to H_T^*(V)$ induced by the restriction of Φ. The morphism is defined at the stalk of hyperedges similarly, but using components of $Y^{\ker(\alpha_0)}$ and $X^{\ker(\alpha_0)}$. This sheaf morphism is natural with respect to the block decomposition from the proof of Lemma 2.5, so $H^0(h) = l$. □

2.3.2 Group actions and induction

Definition 5. We say that a finite group G acts on a GKM-hypergraph $\Gamma = (\mathcal{V}, \mathcal{E}, I, \alpha)$ if G acts on both $\mathcal{V}$ and $\mathcal{E}$, in such a way that I is equivariant and α is invariant. We denote by $GKM(\Gamma)_G$ the category of equivariant GKM-sheaves.

Proposition 2.11. *Suppose that a finite group G acts on a GKM-hypergraph $\Gamma = (\mathcal{V}, \mathcal{E}, I, \alpha)$. Then the quotient*

$$\Gamma/G = (\mathcal{V}/G, \mathcal{E}/G, \bar{I}, \bar{\alpha})$$

is a GKM-hypergraph under the induced maps $\bar{I}$ and $\bar{\alpha}$, and the quotient map

$$\pi : \Gamma \to \Gamma/G$$

is a morphism of GKM-hypergraphs.

Proof. To see that $(\mathcal{V}/G, \mathcal{E}/G, \bar{I}, \bar{\alpha})$ is GKM, we must confirm that if $e_1, e_2 \in \mathcal{E}$ satisfy $\alpha(e_1) = \alpha(e_2)$, then $\bar{I}(G \cdot e_1) \cap \bar{I}(G \cdot e_2) \neq \emptyset$ implies $G \cdot e_1 = G \cdot e_2$.

By definition

$$\bar{I}(G \cdot e_1) = \bigcup_{g \in G} I(g \cdot e_1). \tag{16}$$

Because the sets $\{I(e) | e \in \mathcal{E}, \alpha(e) = \alpha(e_1)\}$ form a partition, $\bigcup_{g \in G} I(g \cdot e_1)$ intersects $\bigcup_{g \in G} I(g \cdot e_2)$ only if $g \cdot e_1 = h \cdot e_2$ for some $g, h \in G$, so we conclude that $G \cdot e_1 = G \cdot e_2$.

The fact that π is a GKM-morphism is clear from (16). □

For any $\mathcal{F} \in GKM(\Gamma)_G$ the pushforward $\pi_*(\mathcal{F})$ is a sheaf of $\mathbb{C}G \otimes A$ modules, so we may decompose into G-isotypical components

$$\pi_*(\mathcal{F}) \cong \bigoplus_{\chi \in \hat{G}} \pi_*(\mathcal{F})^\chi,$$

indexed by the set $\hat{G}$ of irreducible G-representations, where for an open set $U \subset \Gamma/G$

$$\pi_*(\mathcal{F})^\chi(U) := \pi_*(\mathcal{F})(U)^\chi \cong \chi \otimes_{\mathbb{C}} Hom_{G-rep}(\chi, \pi_*(\mathcal{F})(U)).$$

Lemma 2.12. *For each $\chi \in \hat{G}$, the isotypical component $\pi_*(\mathcal{F})^\chi$ is a GKM-sheaf over Γ/G and there is a natural isomorphism*

$$H^0(\mathcal{F})^\chi \cong H^0(\pi_*(\mathcal{F})^\chi).$$

Proof. Everything follows easily from the fact that summands of free A-modules are free A-modules and that G-equivariant maps respect isotypical components. $\square$

An important special case of Lemma 2.12 is the G-invariant subsheaf which we denote $\pi_*(\mathcal{F})^G$.

Lemma 2.13. *Let G be a finite group and let X be a nice $G \times T$-space so that X/G is a nice T-space. Then Γ_X inherits a G-action and there is a natural isomorphism of GKM-sheaves*

$$\mathcal{F}_{X/G} \cong \pi_*(\mathcal{F}_X)^G$$

Proof. This follows pretty directly from the fact that $H_T^*((X/G)^T) \cong H_T^*((X^T)/G) \cong H_T^*(X^T)^G$ and similarly for codimension one subtori $H \subset T$. $\square$

We are now in a position to define the induction functor for equivariant GKM-sheaves. Given a GKM-hypergraph $\Gamma = (\mathcal{V}, \mathcal{E}, I, \alpha)$ and a finite group action G, we may form a G-equivariant GKM-graph denoted by

$$G \times \Gamma = (G \times \mathcal{V}, G \times \mathcal{E}, I', \alpha'),$$

where $I'(g, e) = \{g\} \times I(e)$ and $\alpha'((g, e)) = \alpha(e)$.

If Γ already possesses an H-action and $\phi : H \to G$ is a homomorphism, then $G \times \Gamma$ acquires a $G \times H$-action, with the G factor acting as before and H acting by $h \cdot (g, x) = (g\phi(h^{-1}), h \cdot x)$. We denote the quotient GKM-hypergraph

$$G \times_H \Gamma := (G \times \Gamma)/H$$

which inherits a G-action.

Consider now the diagram of GKM-hypergraph morphisms

$$\begin{tikzcd} & G \times \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \Gamma & & G \times_H \Gamma \end{tikzcd} \tag{17}$$

where p is projection onto the Γ factor and q is the orbit space quotient map. For a GKM-sheaf $\mathcal{F} \in GKM(\Gamma)$ we construct a new sheaf $Ind_H^G(\mathcal{F})$ over $G \times_H \Gamma$ as

$$Ind_H^G(\mathcal{F}) := (q_*(p^*(\mathcal{F})))^H.$$

Note that $Ind_H^G(\mathcal{F})$ depends on the homomorphism $\phi : H \to G$ even though it has been suppressed in the notation.

Theorem 2.14. *The above operation on GKM-sheaves determines a functor, called the* ***induction functor****,*

$$Ind_H^G : GKM(\Gamma)_H \mapsto GKM(G \times_H \Gamma)_G.$$

Proof. Functoriality and G-equivariance follow from standard sheaf theory. The only thing to verify is that $Ind_H^G(\mathcal{F})$ is GKM. The GKM-hypergraph $G \times \Gamma$ is simply a disconnected union of a finite number of copies of Γ and on each copy the restriction of $p^*(\mathcal{F})$ is isomorphic to $\mathcal{F}$, so $p^*(\mathcal{F})$ is a GKM-sheaf. Thus $(q_*(p^*(\mathcal{F})))^H$ is also GKM by Lemma 2.12 and Proposition 2.8. □

The above construction can be modelled at the level of spaces in the following way. Let H be a finite group and let X be a $H \times T$-space. For any homomorphism of finite groups $H \to G$, one may form the "mixed quotient" space $G \times_H X$ equipped with an action by $G \times T$ in analogy with the above construction of $G \times_H \Gamma$.

Proposition 2.15. *There is a natural isomorphism* $Ind_H^G(\mathcal{F}_X) \cong \mathcal{F}_{G \times_H X}$.

Proof. The equality follows without difficulty from the isomorphisms $H_T^*((G \times_H X)^T) \cong H_T^*(G \times X^T)^H$ and similarly for codimension one subtori of T. □

Proposition 2.16. *Let $G_1 \to G_2 \to G_3$ be a diagram of homomorphisms of finite groups and let Γ be G_1-equivariant GKM-hypergraph. There is a natural equivalence of functors*

$$Ind_{G_1}^{G_3} \cong Ind_{G_2}^{G_3} \circ Ind_{G_1}^{G_2}$$

over $G_3 \times_{G_1} \Gamma \cong G_3 \times_{G_2} \times G_2 \times_{G_1} \Gamma$.

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
 & & G_3 \times G_2 \times \Gamma \arrow[dl, "f"'] \arrow[dr, "g"] & & \\
 & G_2 \times \Gamma \arrow[dl] \arrow[dr, "h"] & & G_3 \times G_2 \times_{G_1} \Gamma \arrow[dl, "j"'] \arrow[dr] & \\
\Gamma & & G_2 \times_{G_1} \Gamma & & G_3 \times_{G_1} \Gamma
\end{tikzcd} \tag{18}$$

To prove the result, it suffices to show that j^*h_* and g_*f^* are naturally isomorphic as functors from $GKM(G_2 \times \Gamma)$ to $GKM(G_3 \times G_2 \times_{G_1} \Gamma)$.

Let $\Gamma' = G_2 \times \Gamma$. As a graph $G_3 \times \Gamma'/G_1$ is a disconnected union of copies of Γ'/G_1. For any $\mathcal{F} \in GKM(\Gamma')$, the restrictions of $j^*h_*(\mathcal{F})$ and $g_*f^*(\mathcal{F})$ to each copy of Γ' are canonically isomorphic to $h_*(\mathcal{F})$, so j^*h_* and g_*f^* are naturally isomorphic. □

The following concrete description of the stalks of $Ind_H^G(\mathcal{F})$ is useful.

Lemma 2.17. *Let Γ be a GKM-hypergraph with H-action, let $\mathcal{F} \in GKM(\Gamma)_H$ and let $H \to G$ be a homomorphism of finite groups. For $x \in \Gamma$ a vertex or a hyperedge and $g \in G$, we have a natural isomorphism of stalks $Ind_H^G(\mathcal{F})(U_{[(g,x)]}) \cong \mathcal{F}(U_x)^{H_x}$ such that if $e \in \Gamma$ is an edge and $v \in I(e)$ is an incident vertex then*

$$\begin{tikzcd}
\mathcal{F}(U_e)^{H_e} \arrow[r, "\cong"] \arrow[d, "res"] & Ind_H^G(\mathcal{F})(U_{[(g,e)]}) \arrow[d, "res"] \\
\mathcal{F}(U_v)^{H_v} \arrow[r, "\cong"] & Ind_H^G(\mathcal{F})(U_{[(g,v)]})
\end{tikzcd} \tag{19}$$

commutes, where the vertical arrows are sheaf restriction maps.

Proof. Consider the diagram $\Gamma \xleftarrow{p} G \times \Gamma \xrightarrow{q} G \times_H \Gamma$. By Proposition 2.2, the q-preimage of a basic open set $U_{[g,x]}$ is a disconnected union of basic open sets:

$$q^{-1}(U_{[g,x]}) = \coprod_{y \in q^{-1}([g,x])} U_y.$$

By definition

$$Ind_H^G(\mathcal{F})(U_{[g,x]}) = (\coprod_{y \in q^{-1}([g,x])} p^*(U_y))^H \cong p^*(\mathcal{F})(U_{(g,x)}))^{H_x} = \mathcal{F}(U_x)^{H_x}$$

and these isomorphisms respect restriction maps. □

2.3.3 Exterior product

The next operation is designed to model the operation of cartesian product for T-spaces.

Definition 6. For $i = 1, 2$, let $\Gamma_i = (\mathcal{V}_i, \mathcal{E}_i, I_i, \alpha_i)$ be GKM-hypergraphs. The **product GKM-hypergraph** is the GKM-hypergraph $\Gamma_1 \times \Gamma_2 = (\mathcal{V}, \mathcal{E}, I, \alpha)$ defined by

- $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$,
- $\mathcal{E} = \{(e_1, e_2) \in \mathcal{E}_1 \times \mathcal{E}_2 | \alpha_1(e_1) = \alpha_2(e_2)\}$,
- $I = I_1 \times I_2|_{\mathcal{E}}$ and
- $\alpha((e_1, e_2)) = \alpha_1(e_1) = \alpha_2(e_2)$.

Remark 3. Notice that the projection maps $\pi_i : \Gamma_1 \times \Gamma_2 \to \Gamma_i$ are GKM-morphisms.

Despite the fact that GKM-sheaves are not preserved under pullbacks or interior product, they are preserved the following operation formed by composing these two.

Proposition 2.18. *Let $\mathcal{F}_i$ be a GKM-sheaf on Γ_i for $i = 1, 2$. The* ***exterior product***

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 := \pi_1^*(\mathcal{F}_1) \otimes_A \pi_2^*(\mathcal{F}_2)$$

is a GKM sheaf on $\Gamma = \Gamma_1 \times \Gamma_2$.

Proof. The finiteness and locally free conditions clearly hold. It remains to prove that for any $e \in \mathcal{E}$,

$$res_e : (\mathcal{F}_1 \otimes \mathcal{F}_2)(U_e) \to (\mathcal{F}_1 \otimes \mathcal{F}_2)(I(e))$$

is an isomorphism modulo $\alpha(e)$. If $e = (e_1, e_2)$ then by definition, this map can be identified with

$$res_{e_1} \otimes res_{e_2} : \mathcal{F}_1(U_{e_1}) \otimes \mathcal{F}_2(U_{e_2}) \to \mathcal{F}_1(I(e_1)) \otimes \mathcal{F}_2(I(e_2))$$

and res_{e_i} is an isomorphism modulo $\alpha(e) = \alpha_i(e_i)$ for $i = 1, 2$. These tensor products commute with tensor product by $A[\alpha(e)^{-1}]$ so the result follows. □

Proposition 2.19. *Suppose that X and Y are nice T-spaces and that both $H_T^*(X^{\ker(\alpha_0)})$ and $H_T^*(Y^{\ker(\alpha_0)})$ are free A-modules for all $\alpha_0 \in \mathbb{P}(\Lambda)$ (this holds if $H_T^*(X)$ are $H_T^*(Y)$ are torsion free by Lemma 2.4). Then*

$$\mathcal{F}_X \boxtimes \mathcal{F}_Y \cong \mathcal{F}_{X \times Y}$$

for the diagonal T-action on $X \times Y$.

Proof. It is evident that $(X \times Y)^T = X^T \times Y^T$ and that $X^{\ker(\alpha_0)} \times Y^{\ker(\alpha_0)} = (X \times Y)^{\ker(\alpha_0)}$ for all $\alpha_0 \in \mathbb{P}(\Lambda)$, so the corresponding equalities on connected components produces an isomorphism of GKM graphs $\Gamma_X \times \Gamma_Y \cong \Gamma_{X \times Y}$.

The isomorphism of sheaves is defined through the Kunneth morphisms

$$H_T^*(X^T) \otimes_A H_T^*(Y^T) \to H_T^*(X^T \times Y^T) \tag{20}$$

which is an isomorphism for trivial actions and

$$\kappa : H_T^*(X^{\ker(\alpha_0)}) \otimes_A H_T^*(Y^{\ker(\alpha_0)}) \cong H_T^*((X \times Y)^{\ker(\alpha_0)}) \tag{21}$$

which is also an isomorphism by the following argument.

First observe that the Borel localization theorem implies that 21 is an isomorphism after tensoring by the quotient field of A. Since we are assuming the left-hand side is a free A-module, this means that κ is injective. So it suffices to prove that the dimension over $\mathbb{C}$ of $H_T^*(X^{\ker(\alpha_0)}) \otimes_A H_T^*(Y^{\ker(\alpha_0)})$ is greater than or equal to $H_T^*((X \times Y)^{\ker(\alpha_0)})$ in every degree.

The assumption that $H_T^*(X^{ker(\alpha_0)})$ is a free A-module implies (by a simple Eilenberg-Moore spectral sequence argument) that

$$H_T^*(X^{ker(\alpha_0)}) \cong H^*(X^{ker(\alpha_0)}) \otimes A$$

as a graded A-module, and similarly for $Y^{ker(\alpha_0)}$. Thus

$$H_T^*(X^{\ker(\alpha_0)}) \otimes_A H_T^*(Y^{\ker(\alpha_0)}) = H^*((X \times Y)^{\ker(\alpha_0)}) \otimes A$$

where we have used the Kunneth isomorphism $H^*(X^{\ker(\alpha_0)}) \otimes_C H^*(Y^{\ker(\alpha_0)}) \cong H^*((X \times Y)^{\ker(\alpha_0)})$. The classifying map $ET \times_T (X \times Y) \to BT$ determines a Serre spectral sequence

$$H^*((X \times Y)^{\ker(\alpha_0)}) \otimes A \Rightarrow H_T^*((X \times Y)^{\ker(\alpha_0)})$$

and this implies that in every degree the dimension of the left side is greater or equal to the right side. ∎

Proposition 2.20. *Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are GKM-sheaves. Then the kernel and cokernel of the natural map*

$$\phi : H^0(\mathcal{F}_1) \otimes_A H^0(\mathcal{F}_2) \to H^0(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \tag{22}$$

are torsion with support in codimension greater than one. If additionally $H^0(\mathcal{F}_k)$ is free over A for $k = 1, 2$, then ϕ is an isomorphism.

Proof. We use Lemma 2.7 to describe the global sections functor. Let

$$i^*_{k,\alpha_0} : \mathcal{F}_k(\mathcal{V}_k \cup \mathcal{E}^{\alpha_0}_k) \to \mathcal{F}_k(\mathcal{V}_k)$$

denote the restriction map and let $\Delta \subset \mathbb{P}(\Lambda)$ be the finite set of α_0 for which i^*_{k,α_0} is nonzero for either $k = 1, 2$. The map (22) is equivalent to the functorial map

$$\psi : \Big(\bigcap_{\alpha_0 \in \Delta} \mathrm{im}(i^*_{1,\alpha_0}) \Big) \otimes \Big(\bigcap_{\alpha_0 \in \Delta} \mathrm{im}(i^*_{2,\alpha_0}) \Big) \to \bigcap_{\alpha_0 \in \Delta} \mathrm{im}(i^*_{1,\alpha_0} \otimes i^*_{2,\alpha_0}).$$

Our hypotheses imply that $i^*_{\alpha_0}$ is an injective map between free A-modules and that $i^*_{\alpha_0}$ becomes an isomorphism after localizing to the hyperplane $(\alpha_0)^\perp \subset \mathfrak{t}$. If Δ has cardinality one, then ψ is an isomorphism because everything is free. Since localization commutes with tensor products and finite intersections ([Eis95] §2.2), we deduce that if $x \in \mathfrak{t}$ is annihilated by no more than one element of Δ, then the localization of ψ at x is an isomorphism. Consequently, the support of both $\ker(\psi)$ and $\mathrm{cok}(\psi)$ must lie in the union of codimension two planes $(\alpha_0)^\perp \cap (\alpha_1)^\perp$ where α_0 and α_1 vary over distinct elements of Δ.

Now suppose that $H^0(\mathcal{F}_k)$ is a free A-module for $k = 1, 2$ and let M and N denote the source and target of ψ respectively. Then M is free, so $M \cong A^d$ for some non-negative intger d (ignoring the grading). Because $\ker(\psi)$ is torsion submodule of M, it must be zero and we have a short exact sequence

$$0 \to M \xrightarrow{\psi} N \to \mathrm{cok}(\psi) \to 0. \tag{23}$$

Now suppose for the sake of contradiction that $\mathrm{cok}(\psi)$ is nonzero. By the theory of associated primes ([Eis95], §3.1) there exists some nonzero $y \in \mathrm{cok}(\psi)$ for which the annihilator $\mathrm{ann}(y) = \{a \in A | ay = 0\}$ is a prime ideal. By the result on the support of $\mathrm{cok}(\psi)$, there must exist distinct elements $\alpha_0, \alpha_1 \in \Delta \cap \mathrm{ann}(y)$. We apply the functor $Tor_*(A/\alpha_0 A, -)$ to (23). We know N is torsion free because it injects into $(A^n)^{\otimes m}$, so we obtain an exact sequence

$$0 \to Tor_1(A/\alpha_0 A, \mathrm{cok}(\psi)) \to (A/\alpha_0 A)^d$$

where $y \in Tor_1(A/\alpha_0 A, \mathrm{cok}(\psi)) = \{z \in \mathrm{cok}(\psi) | \alpha_0 z = 0\}$. This implies that $(A/\alpha_0 A)^d$ contains a nonzero element annihilated by α_1 which is a contradiction. Thus $\mathrm{cok}(\psi) = 0$ and ψ is an isomorphism. □

2.3.4 Convolution product

Suppose now that Γ is a GKM-hypergraph equipped with the action of an *abelian* group G which is free and transitive on the set of vertices. Then $\Gamma \times \Gamma$ inherits a $G \times G$ action and we may take the quotient $(\Gamma \times \Gamma)/G$ by the antidiagonal subgroup $\{(g, -g) | g \in G\}$. Observe that this is kernel of the multiplication $Mult : G \times G \to G$, which is a homomorphism because G is abelian.

Lemma 2.21. *There is an isomorphism of GKM-hypergraphs*

$$\phi : \Gamma \cong (\Gamma \times \Gamma)/G,$$

which is canonically defined up to a choice of base vertex in Γ. *Moreover,* ϕ *is* G*-equivariant for the residual* G*-action on* $(\Gamma \times \Gamma)/G$.

Proof. Let $\Gamma = (\mathcal{V}, \mathcal{E}, I, \alpha)$. Choose a base vertex $v_* \in \mathcal{V}$. The GKM conditions ensure that for each $\alpha_0 \in \mathbb{P}(\Lambda)$ there exists a unique hyperedge $e_{\alpha_0} \in \mathcal{E}^{\alpha_0}$ with $v_* \in I(e_{\alpha_0})$. Define a morphism $\phi : \Gamma \to (\Gamma \times \Gamma)/G$ by $\phi(v) = [(v_*, v)]$ for all $v \in \mathcal{V}$ and $\phi(e) = [(e_{\alpha(e)}, e)]$ for all $e \in \mathcal{E}$ (where $[x]$ means the G-orbit containing x). It is easy to see that ϕ is a GKM-hypergraph morphism. We claim that it is an isomorphism.

Notice that every G-orbit in $\mathcal{V} \times \mathcal{V}$ passes exactly once through $\{v_*\} \times \mathcal{V}$, so the restriction of ϕ to $\mathcal{V}$ is a bijection. Since $\mathcal{E}^{\alpha_0}$ describes a partition of $\mathcal{V}$, to prove that ϕ restricts to a bijection between hyper-edges, it suffices to prove that $\mathcal{E}^{\alpha_0}$ and $(\mathcal{E}^{\alpha_0} \times \mathcal{E}^{\alpha_0})/G$ have the same cardinality.

Because G acts transitively on $\mathcal{V}$, it also acts transitively on $\mathcal{E}^{\alpha_0}$. Because G is abelian the stabilizer $G_{e_{\alpha_0}}$ is the common stabilizer of all hyperedges in $\mathcal{E}^{\alpha_0}$ and thus also the common stabilizer of all elements in $\mathcal{E}^{\alpha_0} \times \mathcal{E}^{\alpha_0}$ under the anti-diagonal action. Consequently by the orbit-stabilizer theorem, we have

$$|(\mathcal{E}^{\alpha_0} \times \mathcal{E}^{\alpha_0})/G| = |\mathcal{E}^{\alpha_0} \times \mathcal{E}^{\alpha_0}|/|\mathcal{E}^{\alpha_0}| = |\mathcal{E}^{\alpha_0}|$$

completing the proof. ☐

Definition 7. Let $\pi : \Gamma \times \Gamma \to (\Gamma \times \Gamma)/G \cong \Gamma$ be the quotient map and let $Mult : G \times G \to G$ be the multiplication homomorphism. Given two G-equivariant GKM-sheaves $\mathcal{F}$ and $\mathcal{G} \in GKM(\Gamma)_G$, we define the convolution product $\mathcal{F} * \mathcal{G} \in GKM(\Gamma)_G$ by

$$\mathcal{F} * \mathcal{G} := Ind_{G \times G}^{G}(\mathcal{F} \boxtimes \mathcal{G}).$$

Lemma 2.22. *Let* Γ *and* G *be as above and let* $\mathcal{F}, \mathcal{G} \in GKM(\Gamma)_G$. *For basic open sets* $U_x \subseteq \Gamma$ *there are natural isomorphisms*

$$(\mathcal{F} * \mathcal{G})(U_v) \cong \mathcal{F}(U_v) \otimes_A \mathcal{G}(U_v)$$

at vertices v *and*

$$(\mathcal{F} * \mathcal{G})(U_e) \cong (\mathcal{F}(U_e) \otimes_A \mathcal{G}(U_e))^{G_e}$$

at edges e, *where* G_e *is the stabilizer of* e *under the* G*-action.*

Proof. This follows directly from Lemma 2.17. ☐

The convolution product can be produced at the level of spaces in the following way. Suppose that we have two good T-spaces X and Y, each equipped with G-actions commuting with T, for which the GKM graphs Γ_X and Γ_Y are G-equivariantly isomorphic. If G is abelian and acts freely and transitively on vertices then it is possible to form the mixed quotient $(X \times_G Y) = X \times Y / \sim$, where $\sim$ is the relation $(gx, y) = (x, gy)$. $(X \times_G Y)$ inherits a residual G-action and diagonal T-action.

Proposition 2.23. *If* $\mathcal{F}_X$ *and* $\mathcal{F}_Y$ *are locally free, then* $\mathcal{F}_{(X \times Y)/G} \cong \mathcal{F}_X * \mathcal{F}_Y$.

Proof. This follows from Propositions 2.15 and 2.20. ☐

Proposition 2.24. *Let* $\mathcal{F}, \mathcal{G} \in (GKM(\Gamma)_G, *)$. *If* $H^0(\mathcal{F})$ *and* $H^0(\mathcal{G})$ *are free over* A, *then the isotypical components satisfy* $H^0(\mathcal{F} * \mathcal{G})^\chi = H^0(\mathcal{F})^\chi \otimes_A H^0(\mathcal{G})^\chi$.

Proof. This follows from the equivariance of the Proposition 2.20 for exterior products. ☐

2.3.5 The functors commute

Lemma 2.25. *For $i = 1, 2$ let Γ_i be a GKM-hypergraph with an action by a abelian group G_i which is free and transitive on vertices. The exterior tensor product determines a functor*

$$\boxtimes : GKM(\Gamma_1)_{G_1} \times GKM(\Gamma_2)_{G_2} \to GKM(\Gamma_1 \times \Gamma_2)_{G_1 \times G_2},$$

for which there are natural isomorphisms

$$(\mathcal{F}_1 * \mathcal{G}_1) \boxtimes (\mathcal{F}_2 * \mathcal{G}_2) \cong (\mathcal{F}_1 \boxtimes \mathcal{F}_2) * (\mathcal{G}_1 \boxtimes \mathcal{G}_2) \tag{24}$$

for $\mathcal{F}_i, \mathcal{G}_i \in Sh(\Gamma_i)^{GKM}_{G_i}$.

Proof. First notice that the product operation of graphs is associative and commutative, so $(\Gamma_1 \times \Gamma_1) \times (\Gamma_2 \times \Gamma_2)$ is naturally isomorphic to $(\Gamma_1 \times \Gamma_2) \times (\Gamma_1 \times \Gamma_2)$. Then each side of (24) is an invariant pushforward of the naturally isomorphic GKM-sheaves $(\mathcal{F}_1 \boxtimes \mathcal{G}_1) \boxtimes (\mathcal{F}_2 \boxtimes \mathcal{G}_2) \cong (\mathcal{F}_1 \boxtimes \mathcal{F}_2) \boxtimes (\mathcal{G}_1 \boxtimes \mathcal{G}_2)$. ☐

Proposition 2.26. *For $i = 1, 2$ let Γ_i be a GKM-hypergraph equipped with an action by a finite group H_i and let $H_i \to G_i$ be a homomorphism of finite groups. If $\mathcal{F}_i \in GKM(\Gamma_i)$ then there is a natural isomorphism*

$$Ind_{H_1}^{G_1}(\mathcal{F}_1) \boxtimes Ind_{H_2}^{G_2}(\mathcal{F}_2) \cong Ind_{H_1 \times H_2}^{G_1 \times G_2}(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

Proof. We construct the isomorphism $\phi : Ind_{H_1}^{G_1}(\mathcal{F}_1) \boxtimes Ind_{H_2}^{G_2}(\mathcal{F}_2) \to Ind_{H_1 \times H_2}^{G_1 \times G_2}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ as follows.

Using the identifications of Lemma 2.22, ϕ is defined over the basic open set $U_{[(g_1,x_1)\times(g_2,x_2)]}$ to be the natural map

$$\phi_{[(g_1,x_1)\times(g_2,x_2)]} : (\mathcal{F}_1(U_{x_1}))^{H_{1,x_1}} \otimes (\mathcal{F}_2(U_{x_2}))^{H_{2,x_2}} \to (\mathcal{F}_1(U_{x_1}) \otimes \mathcal{F}_2(U_{x_2}))^{H_{1,x_1} \times H_{2,x_2}}$$

which is an isomorphism because the $\mathcal{F}_i(U_{x_i})$ are free A-modules. The naturality of the identifications in Lemma 2.22 allows us glue together these local isomorphisms into an isomorphism of sheaves. ☐

Proposition 2.27. *Let Γ be GKM-graph equipped with a transitive and vertex-free action by a finite abelian group H, and let $\phi : H \to G$ be a homomorphism of finite abelian groups. Then the induced G-action on $G \times_H \Gamma$ is transitive and vertex-free, and the induction functor $Ind_H^G : GKM(\Gamma) \mapsto GKM(G \times_H \Gamma)$ respects the convolution product,*

$$Ind_H^G(\mathcal{F} * \mathcal{G}) \cong Ind_H^G(\mathcal{F}) * Ind_H^G(\mathcal{G})$$

Proof. This may be deduced from Propositions 2.16 and 2.26 using the sequence of natural isomorphisms

$$\begin{aligned} Ind_H^G(\mathcal{F}) * Ind_H^G(\mathcal{G}) &\cong Ind_{G\times G}^G(Ind_H^G\mathcal{F} \boxtimes Ind_H^G\mathcal{G}) \cong Ind_{G\times G}^G(Ind_{H\times H}^{G\times G}(\mathcal{F} \boxtimes \mathcal{G})) \\ &\cong Ind_H^G(Ind_{H\times H}^H(\mathcal{F} \boxtimes \mathcal{G})) = Ind_H^G(\mathcal{F} * \mathcal{G}) \end{aligned}$$

☐

2.4 Twisted actions

Let W be a finite group. Given a torus T a **twist** is a homomorphism

$$\tau : W \to Aut(T),\ w \mapsto \tau_w.$$

In all examples we consider, W will be a Weyl group acting in the standard way on T. A twist induces actions of W on $\mathbb{P}(\Lambda)$ and on $A = S(\mathfrak{t}^*)$ which we also denote by τ.

Given a twist τ, a **τ-twisted action** of W on a T-space X is a map $\rho : W \times X \to X$ such that

$$\rho(w)(t \cdot x) = \tau_w(t) \cdot \rho(x).$$

A **τ-twisted action on a GKM-hypergraph** $\Gamma = (\mathcal{V}, \mathcal{E}, I, \alpha)$ consists of W-actions on the sets $\mathcal{V}$, $\mathcal{E}$, which are equivariant with respect to I and $\alpha : \mathcal{E} \to \mathbb{P}(\Lambda)$. This differs from an ordinary GKM-action, because we allow W to act nontrivially on $\mathbb{P}(\Lambda)$. Clearly a twisted action on a T-space X induces one on Γ_X.

A **τ-twisted W-action on a GKM sheaf** $\mathcal{F}$ over Γ is a lift of a τ-twisted action ρ of W on Γ to an action $\tilde{\rho}$ of W on $\mathcal{F}$ as a sheaf of $\mathbb{Z}$-graded abelian groups, satisfying the identity $\tilde{\rho}_w(fs) = \tau_w(f)\tilde{\rho}_w(s)$ for all $w \in W$, $f \in A = S(\mathfrak{t}^*)$ and sections s of $\mathcal{F}$. A twisted action on a T-space X induces one on $\mathcal{F}_X$ in the obvious way.

A twisted action on $\mathcal{F}$ determines a decomposition of $H^0(\mathcal{F})$ into W-isotypical components, which are modules over the ring of W-invariants A^W. Particularly interesting is the W-invariant summand $H^0(\mathcal{F})^W$.

Let K be a compact Lie group with maximal torus T such that the normalizer $N_K(T)$ intersects every component of K, and let X be a nice K-space. Restricting to the action of T, we may associate a GKM-hypergraph Γ_X and GKM-sheaf $\mathcal{F}_X$. Since the one-skeleton (X_1, X_0) is preserved by $N_K(T)$, we gain a twisted action of $W = N_K(T)/T$ on Γ_X lifting to $\mathcal{F}_X$. The following proposition is a straight forward consequence of Theorem 2.6 and the isomorphism $H_K^*(X) \cong H_T^*(X)^W$.

Proposition 2.28. *There is a natural morphism $\phi : H_K^*(X) \to H^0(\mathcal{F}_X)^W$ of graded A^W-algebras, with kernel and cokernel supported in codimension at least two in $Spec(A^W) = \mathfrak{t}/W$. In the event that X is equivariantly formal, ϕ is an isomorphism.*

2.5 Examples

2.5.1 Monodromy sheaves

So far every example of a GKM-sheaf has come from a T-space. Now we provide a construction of GKM-sheaves using combinatorial and algebraic data. Let Γ be a GKM-*graph* with a finite number of non-degenerate edges. Let $\mathcal{E}^{nd} \subset \mathcal{E}$ denote the set of nondegenerate edges. For the sake of convenience we choose an orientation for each $e \in \mathcal{E}^{nd}$. That is, we choose source and target maps $s, t : \mathcal{E}^{nd} \to \mathcal{V}$, such that $I(e) = \{s(e), t(e)\}$. A **local system** on Γ consists of a (finitely generated, $\mathbb{Z}$-graded) free A-module M called the **fibre** and a map $\rho : \mathcal{E}^{nd} \to Aut(M)$.

Definition 8. The **monodromy GKM-sheaf** associated to the local system (Γ, M, ρ) is the GKM-sheaf $\mathcal{F} = \mathcal{F}_\rho$ with stalks $\mathcal{F}(U_v) = M$ at vertices, $\mathcal{F}(U_e) = M \oplus M[2]$ at

edges and restriction maps $\mathcal{F}(U_e) = M \oplus M[2] \stackrel{res_e}{\to} M \oplus M = \mathcal{F}(U_{s(e)}) \oplus \mathcal{F}(U_{t(e)})$ by the matrix

$$res_e := \begin{pmatrix} 1 & \alpha(e) \\ \rho(e) & -\alpha(e)\rho(e) \end{pmatrix} \tag{25}$$

where we have abusively used $\alpha(e)$ to denote a generator of the projective weight $\alpha(e) \in \mathbb{P}(\Lambda)$.

This construction produces a GKM-sheaf because the matrix (25) becomes invertible after inverting $\alpha(e)$.

Example 5. The traditional GKM-manifolds of [GZ00] provide the simplest example of a monodromy sheaf. In this case the fibre $M = A$ and all the automorphisms $\rho(e) \in Aut(A)$ are the identity.

2.5.2 GKM manifolds with non-isolated fixed points

We now consider a class of T-manifolds introduced by Guillemin-Holm [GH04] in the context of Hamiltonian actions on symplectic manifolds. By our definition, a **GKM manifold with nonisolated fixed points** is a closed T-manifold X, all of whose fixed point components are homeomorphic to a fixed reference space F, with GKM-graph $\Gamma_X = (\mathcal{V}, \mathcal{E}, I, \alpha)$ such that for each nondegenerate edge E with (necessarily distinct) vertices V_s, V_t, there exists a commutative diagram:

$$\begin{tikzcd} V_s \arrow[r, bend right, "i_s"'] & E \arrow[l, bend right, "\pi_s"'] \arrow[r, bend left, "\pi_t"] & V_t \arrow[l, bend left, "i_t"] \end{tikzcd}$$

where π_s and π_t are T-equivariant S^2-fibre bundles for which the inclusions i_s and i_t are sections.

Proposition 2.29. *If $\mathcal{F}_\rho$ is the monodromy GKM-sheaf on Γ with fibre $H_T^*(F) \cong H_T^*(V)$ and $\rho(E) = (\pi_s \circ i_t)^*$ then $\mathcal{F}_\rho \cong \mathcal{F}_X$.*

Proof. The isomorphisms $\mathcal{F}_\rho(U_V) \cong H_T^*(F) \cong \mathcal{F}_X(U_V)$ at vertices is clear from the definition. The isomorphism

$$\mathcal{F}_\rho(U_E) = H_T^*(F) \oplus H_T^*(F)[2] \cong H_T^*(E) = \mathcal{F}_X(U_E)$$

follows from the Thom isomorphism for the sphere bundle $\pi_s : E \to V_s \cong F$. That the restriction maps match up is an easy exercise. $\square$

Notice that traditional GKM-manifolds are the special case that F is a point and the local system is necessarily trivial. The examples studied by Guillemin and Holm [GH04] also determine trivial local systems. In §3, we show that for regular value $c \in K$, the representation variety $\mathcal{R}_K^1(c)$ is a GKM-manifold with nonisolated fixed points, and that $\mathcal{F}_{\mathcal{R}_K^1(c)}$ is a monodromy sheaf with non-trivial monodromy in general.

2.5.3 Pure Γ-sheaves

In [BM01] Braden-MacPherson introduce the notion of a pure Γ-sheaf $\mathcal{M}$ over a moment graph. They also show that in many interesting cases, the intersection cohomology $IH_T^*(X)$ of a T-equivariant complex variety is equal to the global sections $H^0(\mathcal{M})$ of a pure Γ-sheaf $\mathcal{M}$ associated to X.

Proposition 2.30. *To any pure Γ-sheaf $\mathcal{M}$ there is a canonically associated GKM-sheaf $\mathcal{M}'$ such that $H^0(\mathcal{M}) \cong H^0(\mathcal{M}')$.*

In our framework, a moment graph is essentially the same thing as a GKM-graph $(\mathcal{V}, \mathcal{E}, I, \alpha)$ with only a finite number of non-degenerate edges $\mathcal{E}^{nd} \subset \mathcal{E}$, and an ordering $I(e) = \{v_s, v_t\}$ for $e \in \mathcal{E}^{nd}$ subject to some constraints. A **Γ-sheaf** $\mathcal{M}$ consists of A-modules $\mathcal{M}(v)$ for each vertex and $\mathcal{M}(e)$ for each $e \in \mathcal{E}^{nd}$ and a homomorphism $\rho_{v,e} : \mathcal{M}(v) \to \mathcal{M}(e)$ for every pair $\{(v,e) \in \mathcal{V} \times \mathcal{E}^{nd} | v \in I(e)\}$. The module of global sections

$$H^0(\mathcal{M}) = \{(m_x) \in \bigoplus_{x \in \mathcal{V} \cup \mathcal{E}^{nd}} \mathcal{M}(x) |\ \rho_{v,e}(m_v) = m_e\ \forall \rho_{v,e}\}$$

Remark 4. It is explained in [BM01] that the data defining a Γ-sheaf is equivalent to a sheaf of A-modules over $\mathcal{V} \cup \mathcal{E}$ in the topology dual to the one we use, in the sense that their closed sets are equal to our open sets and vice-versa.

A Γ-sheaf is called **pure** if $\mathcal{M}(v)$ is free for all $v \in \mathcal{V}$ and if for every edge $e \in \mathcal{E}^{nd}$ with $I(e) = \{v_s, v_t\}$ satisfies, $\mathcal{M}(e) \cong \mathcal{M}(v_s)/\alpha(e)\mathcal{M}(v_s)$ with

$$\rho_{v_s,e} : \mathcal{M}(v_s) \to \mathcal{M}(e) = \mathcal{M}(v_s)/\alpha(e)\mathcal{M}(v_s)$$

the projection, plus some additional constraints.

Proof of Proposition 2.30. Let $\mathcal{M}$ be a pure Γ-sheaf. Define a new sheaf of A-modules $\mathcal{M}'$ by

- $\mathcal{M}'(\{v\}) = \mathcal{M}'(U_v) = \mathcal{M}(v)$ for all $v \in \mathcal{V}$,
- for all $e \in \mathcal{E}^{nd}$ the stalk $\mathcal{M}'(U_e)$ and restriction map res_e are defined by the short exact sequence

$$0 \to \mathcal{M}'(U_e) \overset{res_e}{\to} \mathcal{M}(v_s) \oplus \mathcal{M}(v_t) \overset{\rho_{v_s,e}+\rho_{v_t,e}}{\longrightarrow} \mathcal{M}(e) \to 0, \tag{26}$$

- for all degenerate edges $e \in \mathcal{E} \setminus \mathcal{E}^{nd}$, $\mathcal{M}'(e) = \mathcal{M}'(I(e))$ with res_e the identity map.

It is pretty clear that $\mathcal{M}'$ is a sheaf, res_e is an isomorphism modulo $\alpha(e)$ for all $e \in \mathcal{E}$. Both $H^0(\mathcal{M})$ and $H^0(\mathcal{M}')$ inject into $\bigoplus_{v \in \mathcal{V}} \mathcal{M}(v)$ by projection, and the short exact sequence (26) ensures that they have the same image, so $H^0(\mathcal{M}) \cong H^0(\mathcal{M}')$.

It remains to show that $\mathcal{M}'(U_e)$ is a free A-module for every $e \in \mathcal{E}^{nd}$. Because $\mathcal{M}(v_s)$ and $\mathcal{M}(v_t)$ are projective, $\rho_{v_s,e} + \rho_{v_t,e}$ lifts to a map $f : \mathcal{M}(v_s) \oplus \mathcal{M}(v_t) \to \mathcal{M}(v_s)$ and f admits a section. Thus $\ker(f) \cong \mathcal{M}(v_t)$ and $\mathcal{M}'(U_e)$ fits into an exact sequence $0 \to \ker(f) \to \mathcal{M}'(U_e) \to \alpha(e)\mathcal{M}(v_s) \to 0$ which must also split so

$$\mathcal{M}'(U_e) \cong \mathcal{M}(v_t) \oplus \alpha(e)\mathcal{M}(v_s)$$

is free. $\square$

2.5.4 Mutants of compactified representations

These are examples of closed T-manifolds introduced by Franz and Puppe [FP08] whose equivariant cohomology is torsion-free but not free over A. There are three examples: Z_r, $r = 2, 4, 8$ with action by $T = (U(1))^{r+1}$, whose construction makes use of the Hopf fibration $S^{r-1} \to S^{2r-1} \to S^r$. Franz and Puppe prove that as graded A-modules

$$H_T^*(Z_r) \cong A \oplus m[r-1] \oplus A[2r+2] \oplus A[3r+1]$$

where m is the augmentation ideal of A.

The GKM-sheaf $\mathcal{F}_{Z_r}$ is easily determined from the description of Z_r in [FP08], which we do not reproduce it here. The GKM-hypergraph Γ_{Z_r} consists of two vertices and $r+1$ non-degenerate edges labelled by distinct weights $\alpha_0, ..., \alpha_r$ which form a basis of $\mathfrak{t}^*$. The GKM-sheaf $\mathcal{F}_{Z_r}$ is the trivial monodromy sheaf with fibres $M := H^*(S^{r-1}) \otimes A \cong A \oplus A[r-1]$. It follows that the global sections of $\mathcal{F}_{Z_r}$ may be identified with the image of the matrix

$$\begin{pmatrix} 1 & f \\ 1 & -f \end{pmatrix}$$

in $M \oplus M$, where $f = \prod_{i=0}^r \alpha_i$ has degree $2(r+1)$. Thus

$$H^0(\mathcal{F}_{Z_r}) \cong A \oplus A[r-1] \oplus A[2r+2] \oplus A[3r+1].$$

This example shows that $H^0(\mathcal{F}_X)$ may be a free A-module even when $H_T^*(X)$ is not.

2.5.5 Equivariant de Rham theory and graphs

Given a GKM-graph Γ, consider the trivial monodromy sheaf $\mathcal{F}$ over Γ with fibre A. The **cohomology of the graph** $H_T(\Gamma)$ is defined to equal $H^0(\mathcal{F})$. It is interesting to ask under what circumstances $H_T(\Gamma)$ is a free A-module and what its Betti numbers are.

In a series of papers, [GZ00, GZ01, GZ03] Guillemin and Zara translated concepts from Hamiltonian actions on symplectic manifolds to GKM-theory, motivated in part by these questions. They a define the notion of moment map on a GKM-graph Γ, define the reduction $\Gamma//S^1$ with respect to such a moment map and prove a version of Kirwan surjectivity $\kappa : H_T(\Gamma) \to H_{T/S^1}(\Gamma//S^1)$.

In certain circumstances, the reduction $\Gamma//S^1$ is a a GKM-*hyper*graph but not a graph and in this situation, addressed in [GZ03], the arguments become rather technical. We believe the reduction process could be more clearly understood in our framework (we do not pursue this in this paper). For example the definition of $H_T(\Gamma//S^1)$ "by duality" is a strong hint that our topology, which is dual to the more obvious one in Braden-MacPherson [BM01] may be important (see Remark 4). Also, the locality of $H_T(\Gamma)$ would become manifest if it were defined as the global sections of a GKM-sheaf.

3 Representation varieties for a punctured surface

It will be useful to introduce a larger class of representation varieties than those defined in the introduction. Gauge theoretically, they correspond to moduli spaces of flat bundles over a punctured nonorientable surface with prescribed holonomy around the puncture.

Let Σ_g denote the connected sum of $g+1$[5] copies of $\mathbb{R}P^2$. By the classification of compact surfaces, every nonorientable compact surface without boundary is isomorphic to Σ_g for some $g = 0, 1, 2, ...$ The fundamental group $\pi_1(\Sigma_g)$ has presentation

$$\pi_1(\Sigma_g) \cong \{a_0, ..., a_g | \prod_{i=0}^{g} a_i^2 = \mathbb{1}\}. \tag{27}$$

Let Γ_g denote the free group on $g+1$ generators $\{a_0, ..., a_g\}$ so that the presentation (27) determines a surjection $\Gamma_g \to \pi_1(\Sigma_g)$. For K a compact connected Lie group and $c \in K$, define

$$\mathcal{R}_K^g(c) := \{\phi \in Hom(\Gamma_g, K) |\ \phi(\prod_{i=0}^{g} a_i^2) = c\}.$$

This is a topological space under the compact open topology with conjugation action by the centralizer $Z_K(c) \subset K$. In fact, the embedding $\mathcal{R}_K^g(c) \hookrightarrow K^{g+1}$ sending ϕ to $(\phi(a_0), ..., \phi(a_g))$ identifies $\mathcal{R}_K^g(c)$ with the compact real algebraic set

$$\mathcal{R}_K^g(c) \cong \{(k_0, ..., k_g) \in K^{g+1} | \prod_{i=0}^{g} k_i^2 = c\} \tag{28}$$

for which the $Z_K(c)$-action is algebraic. Notice that

$$\mathcal{R}_K^g(\mathbb{1}) \cong Hom(\pi_1(\Sigma_g), K)$$

where $\mathbb{1} \in K$ is the identity, so we recover the representation varieties described in the introduction.

We will always choose c to lie in a chosen maximal torus $T \subset K$, with complexified lie algebra $\mathfrak{t}$. We use notation $W := N_K(T)/T$ for the Weyl group, and $W_c := N_{Z_K(c)}(T)/T$ for the portion of the Weyl group that stabilizes c. The well known isomorphism

$$H^*_{Z(c)}(\mathcal{R}_K^g(c)) \cong H_T^*(\mathcal{R}_K^g(c))^{W_c}$$

allows us to divide the study of general compact group actions into torus actions and Weyl group actions. As before, we use notation $A := \mathbb{C}[\mathfrak{t}] = S(\mathfrak{t}^*) \cong H^*(BT)$.

3.1 Fixed points

Our first task is to describe the fixed points of $\mathcal{R}_K^g(c)$ under the conjugation action by T. Maximal tori are maximal abelian, so it follows that a homomorphism $\phi \in \mathcal{R}_K^g(c)$ is fixed by T if and only if $\mathrm{im}(\phi) \subset T$, thus

$$\mathcal{R}_K^g(c)^T = \mathcal{R}_T^g(c).$$

For this reason, it is useful to describe with some care the case that $K = T = U(1)^r$ is a torus.

[5]The index is chosen so that g is the genus of the orientable double cover.

Any homomorphism from $\pi := \pi_1(\Sigma_g)$ to an abelian group must factor through the abelianization $\pi/[\pi, \pi] \cong H_1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^g \oplus \mathbb{Z}_2$ so we obtain

$$\mathcal{R}^g_T(\mathbb{1}) = Hom(\pi, T) \cong Hom(H_1(\Sigma_g, \mathbb{Z}), T) \cong T^g \times T_2. \tag{29}$$

where $T_2 = Tor_1^{\mathbb{Z}}(\mathbb{Z}/2, T) \cong (\mathbb{Z}/2)^r$ is the 2-torsion subgroup of T. More canonically, regarding $\mathcal{R}^g_T(\mathbb{1})$ as a group under pointwise multiplication, we have a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & T^g \arrow[r] & \mathcal{R}^g_T(\mathbb{1}) \arrow[l, bend right, "\rho"'] \arrow[r] & T_2 \arrow[r] & 0 \end{tikzcd} \tag{30}$$

and we choose a splitting $\rho : \mathcal{R}^g_T(\mathbb{1}) \to T^g$, $\phi \mapsto \phi(a_1, ..., a_g)$, using the presentation (27) of the fundamental group.

For general $c \in T$, pointwise multiplication determines a free and transitive action of $\mathcal{R}^g_T(\mathbb{1})$ on $\mathcal{R}^g_T(c)$, i.e.

Lemma 3.1. *For $g \geq 0$, T a rank r compact torus and $c \in T$, the representation variety $\mathcal{R}^g_T(c)$ is a torsor for $\mathcal{R}^g_T(\mathbb{1})$, thus is diffeomorphic to $T^g \times T_2$.*

Remark 5. The action of T_2 on $\mathcal{R}^g_T(c)$ is canonically defined up to isotopy, and acts freely and transitively on the set of components. Thus we have a canonical isomorphism $H^*_T(F_i) \cong H^*_T(F_j)$ for every pair of connected components F_i, F_j of $\mathcal{R}^g_T(c)$. Less canonically, they may all be identified with $H_T(T^g) \cong (\wedge \mathfrak{t}^*)^{\otimes g} \otimes A$, with all tensors taken over $\mathbb{C}$.

Corollary 3.2. *For any compact, connected Lie group K of rank r, the GKM-hypergraph of $\mathcal{R}^g_K(c)$ has a vertex set $\mathcal{V}$ consisting of 2^r vertices which are naturally identified with the set of square roots of c,*

$$\mathcal{V} \cong \{t \in T | t^2 = c\}$$

upon which T_2 acts freely and transitively by multiplication. The stalks of the GKM-sheaf $\mathcal{R}^g_K(c)$ at vertices may be identified with $\wedge(\mathfrak{t}^*)^{\otimes g} \otimes A$ and the T_2-action preserves these identifications.

Because T_2 acts freely on the vertex set $\mathcal{V}$, we gain a convolution product $*$ on T_2-equivariant GKM-sheaves.

Lemma 3.3. *Let $\mathcal{F}_g := \mathcal{F}_{\mathcal{R}^g_T(c)}$. Then*

$$\mathcal{F}_g \cong \mathcal{F}_1 * ... * \mathcal{F}_1 = (\mathcal{F}_1)^{*g}$$

Proof. We begin with the case $c = \mathbb{1}$ which we prove by induction. We use the characterization (28)

$$\mathcal{R}^g_K(\mathbb{1}) = \{(t_0, ..., t_g) \in T^{g+1} | t_0^2 ... t_g^2 = \mathbb{1}\}$$

and T_2 acts by multiplying t_0. By Proposition 2.23 the induction step holds by the T_2-equivariant homeomorphism

$$\psi : \mathcal{R}^1_K(\mathbb{1}) \times_{T_2} \mathcal{R}^g_K(\mathbb{1}) \cong \mathcal{R}^{g+1}_K(\mathbb{1})$$

where $\psi((s_0, s_1) \times (t_0, ..., t_g)) = (s_0 t_0, s_1, t_1, t_2, ..., t_g)$.

The case of general $c \in T$ follows by the fact that $\mathcal{R}^g_K(c)$ is a torsor for $\mathcal{R}^g_K(\mathbb{1})$, thus T_2-equivariantly diffeomorphic. $\square$

Remark 6. For $c \neq \mathbb{1}$, the isomorphism $\mathcal{F}_g \cong \mathcal{F}_1 * ... * \mathcal{F}_1$ of Lemma 3.3 has only been defined up to an automorphism of T_2, or equivalently up to a choice of base vertex. In future sections, we will prove that for compact, connected K and $c \in T \subset K$ a *regular* element, the T_2-action extends to the full one-skeleton of $\mathcal{R}^g_K(c)$, so this ambiguity is not so important. However, when c is not regular, the one-skeleton is usually not T_2-equivariant, and greater care must be taken. The following Lemma is meant to address this.

Lemma 3.4. *A continuous path $\gamma : [0,1] \to T$ induces an isotopy class of T_2-equivariant homeomorphisms $\mathcal{R}^g_T(\gamma(0)) \to \mathcal{R}^g_T(\gamma(1))$ (where T_2 acts as in Lemma 3.1). This class contains a map sending $(t_0, ..., t_g)$ to $(st_0, t_1, ..., t_g)$, where $s \in T$ satisfies $s^2 = \gamma(0)^{-1}\gamma(1)$.*

Proof. The variety $\mathcal{R}^g_T(c)$ is the fibre at $c \in T$ of the submersion $T^{g+1} \to T$ sending $(t_0, ..., t_g)$ to $\prod_{i=0}^g t_i^2$. Pulling back by γ determines a fibre bundle $\gamma^* T^{g+1}$ over $[0,1]$ inducing a homeomorphism up to homotopy between fibres $\mathcal{R}^g_T(\gamma(0))$ and $\mathcal{R}^g_T(\gamma(1))$.

The T_2-action is induced by a canonical $\mathcal{R}^g_T(\mathbb{1})$-action. That the homeomorphism can be made equivariant follows from the classical result of Palais-Stewart [PS60] on the rigidity of compact group actions on compact manifolds.

Define the path $\delta : [0,1] \to T$ by $\delta(x) = \gamma(0)^{-1}\gamma(x)$. Because the squaring map is a covering, there is a unique path $\sqrt{\delta} : [0,1] \to T$ such that $(\sqrt{\delta}(x))^2 = \delta(x)$ and $\sqrt{\delta}(0) = \mathbb{1}$. Define a T_2-equivariant bundle trivialization $\phi : [0,1] \times \mathcal{R}^g_T(\gamma(0)) \to \gamma^* T^{g+1}$ by $\phi(x, (t_0, ..., t_g)) = (x, (\sqrt{\delta}(x)t_0, ..., t_g))$. Setting $s = \sqrt{\delta}(1)$ completes the proof. ◻

Corollary 3.5. *Let $\gamma : [0,1] \to T$ be a continuous path and denote $\mathcal{R}_i = \mathcal{R}^g_K(\gamma(i))$ for $i = 0, 1$. Then γ determines a T_2-equivariant bijection between the vertex sets $\mathcal{V}_0 \cong \mathcal{V}_1$ and this lifts to a T_2-equivariant isomorphism of restricted sheaves*

$$\mathcal{F}_{\mathcal{R}_0}|_{\mathcal{V}_0} \cong \mathcal{F}_{\mathcal{R}_1}|_{\mathcal{V}_1}.$$

3.2 The case K=SU(2)

We review here the main results from [Bai10], using the language of GKM-sheaves. Throughout §3.2, set $K = SU(2)$, and $T \subset K$ is a maximal torus with $c \in T$. The centre $Z(K)$ consists of $\pm\mathbb{1} = T_2$, and all other values of c are regular.

Theorem 3.6 ([Bai10] Thm 1.2)**.** *The representation varieties $\mathcal{R}^g_{SU(2)}(c)$ are equivariantly formal under conjugation by T and by $Z(c)$ for all $c \in K$ and $g \in \{0, 1, 2, ...\}$. This means in particular that $H^*_T(\mathcal{R}^g_K(c)) \cong H^0(\mathcal{F}_{\mathcal{R}^g_K(c)})$ is free over A.*

Since T has rank one, $\mathbb{P}(\Lambda)$ is a single point.

Proposition 3.7. *The GKM-hypergraph $\Gamma_{\mathcal{R}^g_{SU(2)}(c)}$ for any $c \in T$ and $g \geq 1$ consists of two vertices and a single edge e connecting them, with $\alpha(e)$ equal to the sole element of $\mathbb{P}(\Lambda)$.*

Proof. This follows from Corollary 3.2 and the fact the $\mathcal{R}^g_{SU(2)}(c)$ is connected. ◻

The case when $g = 1$ and c is regular can be described quite explicitly. In this case $\mathcal{R}^1_K(c)$ is diffeomorphic to $S^1 \times S^2$ and T acts via rotation on S^2 with weight 2. In particular:

Proposition 3.8. *For regular c, $\mathcal{R}^1_K(c)$ is a GKM-manifold with non-isolated fixed points. The local system has fibre $H^*_T(T) \cong \wedge\mathfrak{t}^* \otimes A$, with holonomy map $\rho(e) = S_{\alpha(e)} \otimes Id_A$, where $S_{\alpha(e)}$ is the automorphism of $\wedge(\mathfrak{t}^*)$ induced by multiplication by -1 on $\mathfrak{t}$, which we think of as reflection in the root hyperplane $\alpha(e)^\perp = \{0\}$.*

Proof. That $\mathcal{R}^1_K(c)$ is a GKM-manifold with non-isolated fixed points is clear from the preceding description. The fibres were described in Remark 5. The holonomy map can be inferred from Propositions 5.3 and 5.4 of [Bai10]. □

Remark 7. Notice that because the graph $\Gamma_{\mathcal{R}^1_K(c)}$ is "simply connected", the local system above can be trivialized. However the convention adopted in Remark 5 of using the T_2-action to identify fixed point components forces the local system to be non-trivial.

Because $T_2 = \{\pm\mathbb{1}\}$ lies in the centre of K, the T_2-action on $\mathcal{R}^g_T(c)$ described in Remark 5 extends to $\mathcal{R}^g_T(c)$ in the obvious way. This makes $\mathcal{F}_{\mathcal{R}^g_K(c)}$ into a T_2-equivariant GKM-sheaf.

Proposition 3.9. *Let $c \in T$ be a regular element and let $\mathcal{F}_g := \mathcal{F}_{\mathcal{R}^g_K(c)}$. For $g \geq 1$ we have an isomorphism between the GKM-sheaves $\mathcal{F}_g \cong \mathcal{F}_1 * ... * \mathcal{F}_1 = (\mathcal{F}_1)^{*g}$.*

Proof. Follows from Propositions 5.3 and 5.4 of [Bai10]. □

Now we turn to the non-regular cases $\epsilon \in \{\pm\mathbb{1}\}$. The full group K centralizes ϵ and so K acts by conjugation on $\mathcal{R}^g_K(\epsilon)$, and $\mathcal{F}_{\mathcal{R}^g_K(\epsilon)}$ acquires a twisted $W_\epsilon = W$-action as described in §2.4. Choose a path $\gamma : [0,1] \to K$ connecting ϵ with some regular element c. Using Lemma 3.4 we obtain a twisted action of W on the restriction of $\mathcal{F}_{\mathcal{R}^g_T(c)}$ to the vertex set.

Proposition 3.10. *The action described above extends to a twisted action of the Weyl group W on $\mathcal{F}_{\mathcal{R}^g_{SU(2)}(c)}$, twisted by the standard action of W on $\mathfrak{t}$. Taking W-invariants produces an isomorphism*

$$H^*_K\big(\mathcal{R}^g_K(\epsilon)\big) \cong H^0(\mathcal{F}_{\mathcal{R}^g_K(c)})^W.$$

Proof. Using (29) to identify $\mathcal{R}^g_T(\epsilon) \cong T^g \times T_2$, the action of the nontrivial element $w \in W$, sends $(t_1, ..., t_g, z)$ to $(wt_1w, ..., wt_gw, \epsilon z)$. The result now follows from the explicit description of the image of the localization map $H^*_T(\mathcal{R}^g_K(\epsilon)) \to H^*_T(\mathcal{R}^g_T(\epsilon))$ found in Propositions 5.3, 5.4 and 5.5 from [Bai10]. □

Remark 8. This result is stranger than it might first appear. The W-action on $\mathcal{F}_{\mathcal{R}^g_K(c)}$ is *not* in general induced by one on $\mathcal{R}^g_K(c)$. Moreover, while the result implies that $H^*_T\big(\mathcal{R}^g_K(\epsilon)\big)^W \cong H^0(\mathcal{F}_{\mathcal{R}^g_K(c)})^W$, it is *not* true in general that $H^*_T\big(\mathcal{R}^g_K(\epsilon)\big) \cong H^0(\mathcal{F}_{\mathcal{R}^g_K(c)})$.

Remark 9. It will be important later to observe that both $\epsilon \in \{\pm\mathbb{1}\}$, the action of W on $\mathcal{F}_g$ described in Proposition 3.10 commutes with the T_2-action, and in fact differ by the non-trivial element of T_2.

3.3 K has semisimple rank one

Let K be a compact connected Lie group of rank r with complexified Lie algebra $\mathfrak{k}$ and let $T \subset K$ be a maximal torus with complexified Lie algebra $\mathfrak{t}$. We may decompose $\mathfrak{k}$ into its central and semisimple parts:

$$\mathfrak{k} = Z(\mathfrak{k}) \oplus \mathfrak{k}_{ss}$$

In this section, we consider the case where $\mathfrak{k}_{ss}$ has rank 1 or equivalently, $\mathfrak{k}_{ss} \cong \mathfrak{su}(2) \otimes \mathbb{C}$. The following lemma is elementary.

Lemma 3.11. *If K has semisimple rank one, then it is isomorphic to one of the following:*
(i) $U(1)^{r-1} \times SU(2)$
(ii) $U(1)^{r-2} \times U(2)$
(iii) $U(1)^{r-1} \times SO(3)$

If K has form (ii) or (iii), then it fits into a short exact sequence

$$0 \to C_2 \to \tilde{K} \xrightarrow{\phi} K \to 1, \tag{31}$$

where $\tilde{K} = U(1)^{r-1} \times SU(2)$ is of type (i) and $C_2 \cong \mathbb{Z}/2\mathbb{Z}$. A recurring strategy in this section is to reduce cases (ii) and (iii) to case (i) using (31), which is in turn reduced to the well understood cases $SU(2)$ and $U(1)$. For $c \in K$, we denote by $\tilde{c}$ an element of $\tilde{K}$ satisfying $\phi(\tilde{c}) = c$.

Proposition 3.12. *For all $g \geq 0$ and $c \in T$ the representation variety $\mathcal{R} := \mathcal{R}^g_K(c)$ is equivariantly formal, so $H^0(\mathcal{F}_{\mathcal{R}}) \cong H^*_T(\mathcal{R})$ by Theorem 2.6.*

Proof. If $K = SU(2) \times U(1)^{r-1}$ and $c = (c_1, c_2)$, then $\mathcal{R}^g_K(c) = \mathcal{R}^g_{SU(2)}(c_1) \times \mathcal{R}^g_{U(1)^{r-1}}(c_2)$ and the result follows from the $SU(2)$ case because the product of equivariantly formal spaces is equivariantly formal under the product group action.

Otherwise, K fits into a short exact sequence (31). Then $\mathcal{R}^g_{\tilde{K}}(\tilde{c})$ is equivariantly formal and is acted upon freely by the finite group $Hom(\pi, C_2) \cong C_2^{g+1}$ so that ϕ determines an isomorphism

$$\mathcal{R}^g_K(c) \cong \coprod_{\phi(\tilde{c})=c} \mathcal{R}^g_{\tilde{K}}(\tilde{c})/C_2^{g+1}. \tag{32}$$

Applying equivariant cohomology

$$H^*_T(\mathcal{R}^g_K(c)) \cong H^*_{\tilde{T}}(\mathcal{R}^g_K(c)) \cong \bigoplus_{\phi(\tilde{c})=c} H^*_{\tilde{T}}(\mathcal{R}^g_{\tilde{K}}(\tilde{c}))^{C_2^{g+1}}$$

we have $H^*_T(\mathcal{R}^g_K(c))$ is a summand of a free A-module, hence free. $\square$

Remark 10. There is a small subtlety in the above proof that should be explained. We are working with complex coefficients so the covering $\tilde{T} \to T$ induces an isomorphism $H^*_{\tilde{T}}(\mathcal{R}^g_{\tilde{K}}(\tilde{c})) \cong H^*_T(\mathcal{R}^g_{\tilde{K}}(\tilde{c}))$ so both actions are formal.

We are left with the problem of understanding the GKM-sheaf $\mathcal{F}_{\mathcal{R}}$. We begin with determining the GKM-hypergraph.

Observe that since K has semisimple of rank one, the pair (K, T) has a unique pair of roots, that we denote $\pm\alpha_0$. Every point of $\mathcal{R}^g_K(c)$ is fixed by $\ker(\alpha_0)$, so the one-skeleton is the whole space.

Define the co-root $h_{\alpha_0} \in \mathfrak{t}$ to be the unique element in $\mathfrak{t} \cap \mathfrak{k}_{ss}$ satisfying $\alpha_0(h_{\alpha_0}) = 2$. The exponential $exp(2\pi i h_{\alpha_0}) = \mathbb{1} \in K$, so $exp(\pi i h_{\alpha_0}) \in T_2$.

Proposition 3.13. *For any $c \in T$ and $g \geq 1$, denote the GKM-hypergraph $\Gamma_{\mathcal{R}^g_K(c)} = (\mathcal{V}, \mathcal{E}, I, \alpha)$. The vertex set $\mathcal{V}$ is naturally identified with the T_2-torsor*

$$\mathcal{V} \cong \{t \in T | t^2 = c\}$$

as explained in Corollary 3.2.

If $K \cong U(1)^{r-1} \times SO(3)$ then all edges are degenerate. Otherwise, the non-degenerate edges $e \in \mathcal{E}$ are those that satisfy $\alpha(e) = \alpha_0$, and these join all pairs of the form

$$\{v, exp(\pi i h_{\alpha_0})v\}. \tag{33}$$

Proof. If $K \cong \tilde{K} \cong SU(2) \times U(1)^{r-1}$, then $\mathcal{R}^g_K(c) = \mathcal{R}^g_{SU(2)}(c_1) \times \mathcal{R}^g_{U(1)^{r-1}}(c_2)$ and the result follows easily. Otherwise, $\Gamma_{\mathcal{R}^g_K(c)}$ is the quotient of a C_2^{g+1} action on $\coprod_{\phi(\tilde{c})=c} \Gamma_{\mathcal{R}^g_K(c)}$. Thus $\mathcal{R}^g_K(c)$ is either degenerate or isomorphic to $\Gamma_{\mathcal{R}^g_{\tilde{K}}(c)}$ depending on if $\mathcal{R}^g_K(c)$ has 2^r or 2^{r-1} connected components. This can be verified directly or using Ho-Liu [HL05]. ☐

Remark 11. When $K \cong U(1)^{r-1} \times SO(3)$, we have $exp(\pi i h_{\alpha_0}) = \mathbb{1}$ so formula (33) correctly describes the degenerate GKM-graph $\Gamma_{\mathcal{R}^g_K(c)}$ in this case.

3.3.1 c is regular

From the description of the GKM-graph $\Gamma_{\mathcal{R}^g_K(c)}$ in Proposition 3.13, it is clear that the free and transitive T_2-action on $\mathcal{V}$ extends to full GKM-graph.

Lemma 3.14. *If $c \in T \subset K$ is regular, then the T_2-action on $\mathcal{R}^g_K(c)^T = \mathcal{R}^g_T(c)$ described in Remark 5 extends to a T-equivariant T_2-action on $\mathcal{R}^g_K(c)$. Consequently, $\mathcal{F}_{\mathcal{R}^g_K(c)}$ is a T_2-equivariant GKM-sheaf.*

Proof. Denote $X := \mathcal{R}^g_K(c)$. If $K = \tilde{K} \cong SU(2) \times U(1)^{r-1}$, then T_2 lies in the centre of K so the action extends to X simply by multiplying the zeroth entry.

Otherwise K fits into the short exact sequence (31). Let $\tilde{c} \in \tilde{K}$ satisfy $\phi(\tilde{c}) = c$ and let $\kappa \in C_2 \subset \tilde{K}$ denote the generator of C_2. It was shown in the proof of Proposition 3.12 that there is a Galois covering map $\tilde{X} \to X$ with Galois covering group C_2^{g+1}, where

$$\tilde{X} := \mathcal{R}^g_{\tilde{K}}(\tilde{c}) \coprod \mathcal{R}^g_{\tilde{K}}(\kappa\tilde{c}).$$

We will construct an action

$$\rho : G \times \tilde{X} \to \tilde{X}$$

of a finite group G on $\tilde{X}$ which descends to the desired action of T_2 on X.

Let $\tilde{T} \subset \tilde{K}$ denote the maximal torus satisfying $\phi(\tilde{T}) = T$ and define $G := \phi^{-1}(T_2) \cap \tilde{T}$. The 2-torsion subgroup $\tilde{T}_2 \subset \tilde{T}$ sits inside G as an index 2 subgroup and the restriction ρ to $\tilde{T}_2$ acts by multiplying the zeroth factor as described above. Now choose $\sqrt{\kappa} \in \tilde{T}_2$ so that $(\sqrt{\kappa})^2 = \kappa$. Since G is generated by $\tilde{T}_2$ and $\sqrt{\kappa}$, to define ρ it suffices to define $\rho(\sqrt{\kappa})$.

Consider now the product of squares map

$$\psi : \tilde{K}^{g+1} \to \tilde{K}, \quad \psi(k_0, ..., k_g) = \prod_{i=0}^{g} k_i^2.$$

By ([Bai10] Prop. 4.1), we know that the set of regular values $\tilde{K}_{\psi\ reg} = (SU(2) \setminus (-\mathbb{1})^{g+1}) \times U(1)^{r-1}$. In particular, $\tilde{T}_{\psi-reg} = \tilde{T} \cap \tilde{K}_{\psi-reg}$ is path connected. Using the same argument as Lemma 3.4, we obtain homeomorphism

$$\Xi : \mathcal{R}^g_{\tilde{K}}(\tilde{c}) \to \mathcal{R}^g_{\tilde{K}}(\kappa\tilde{c})$$

determined up to $\tilde{T} \times \tilde{T}_2 \times C_2^{g+1}$ equivariant isotopy. Define $\rho(\sqrt{\kappa})$ by $\rho(\sqrt{\kappa})|_{\mathcal{R}^g_{\tilde{K}}(\tilde{c})} := \Xi$ and $\rho(\sqrt{\kappa})|_{\mathcal{R}^g_{\tilde{K}}(\kappa\tilde{c})} := \Xi^{-1} \circ \rho(\kappa)$.

Clearly ρ is a well defined action of G on $\tilde{X}$ which commutes with T and C_2^{g+1} and thus descends to an action of T_2 on X. That the restricted T_2-action on X^T is the correct one up to equivariant homotopy follows from Lemma 3.4. To obtain the correct T_2-action on the nose use the equivariant homotopy extension property of ([MPC96], Theorem 3.1). □

For the next string of results we use the following notation. Let c be generic, let $\mathcal{F}_g := \mathcal{F}_{\mathcal{R}^g_K(c)}$ and $\tilde{\mathcal{F}}_g := \mathcal{F}_{\mathcal{R}^g_{\tilde{K}}(\tilde{c})}$ where $\tilde{K} = U(1)^{r-1} \times SU(2)$. There is a covering homomorphism $\tilde{K} \to K$ which is an isomorphism if K is of type (i) and a two-fold cover if K is of type (ii) or (iii). The covering restricts to a homomorphism $\tilde{T}_2 \to T_2$.

Lemma 3.15. *The induction functor*

$$Ind_{\tilde{T}_2}^{T_2} : GKM(\tilde{\Gamma})_{\tilde{T}_2} \mapsto GKM(\Gamma)_{T_2}$$

satisfies $Ind_{\tilde{T}_2}^{T_2}(\tilde{\mathcal{F}}_g) \cong \mathcal{F}_g$.

Proof. If $K \cong \tilde{K}$ then $Ind_{\tilde{T}_2}^{T_2}$ is the identity functor so the assertion is obvious.

Otherwise K fits into the long exact sequence (31). The proof of Lemma 3.14 demonstrates that $\mathcal{R}^g_K(c)$ is constructed from $\mathcal{R}^g_{\tilde{K}}(\tilde{c})$ using the mixed quotient construction associated to the homomorphism $\tilde{T}_2 \to T_2$, so the result follows from Proposition 2.15. □

Proposition 3.16. *Denote $\mathcal{F}_g = \mathcal{F}_{\mathcal{R}^g_K(c)}$ where $c \in K$ is regular. Then $\mathcal{F}_g$ is isomorphic as a GKM-sheaf, with the g-fold convolution product $\mathcal{F}_1 * ... * \mathcal{F}_1$.*

Proof. If $\tilde{K} \cong K \cong U(1)^{r-1} \times SU(2)$ then $\mathcal{R}^g_{\tilde{K}}(c) = \mathcal{R}^g_{SU(2)}(c_1) \times \mathcal{R}^g_{U(1)^{r-1}}(c_2)$ and the result follows from Lemmas 3.3 and 2.25 and Proposition 3.9.

The result for general K follows from the natural isomorphisms

$$\mathcal{F}_1 * ... * \mathcal{F}_1 \cong Ind_{\tilde{T}_2}^{T_2}(\tilde{\mathcal{F}}_1 * .... * \tilde{\mathcal{F}}_1) \cong Ind_{\tilde{T}_2}^{T_2}(\tilde{\mathcal{F}}_g) \cong \mathcal{F}_g$$

applying from Proposition 2.27 and Lemma 3.15. $\square$

Because $\mathcal{F}_g$ is T_2-equivariant, it is possible to decompose $H^0(\mathcal{F}_g)$ into isotypical components

$$H^0(\mathcal{F}_g) \cong \bigoplus_{\chi \in \hat{T}_2} H^0(\mathcal{F}_g)^\chi \cong \bigoplus_{\chi \in \hat{T}_2} H^0(\mathcal{F}_g^\chi),$$

where $\mathcal{F}_g^\chi$ is a GKM-sheaf over the GKM-graph with one vertex.

Lemma 3.17. *The isotypical components satisfy*

$$H^0(\mathcal{F}_g^\chi) \cong H^0(\mathcal{F}_1^\chi)^{\otimes g}$$

where $\mathcal{F}_1^\chi$ is the χ-isotypical pushforward of $\mathcal{F}_1$ to the GKM-graph with one vertex.

Proof. By Proposition 3.16 we have

$$\mathcal{F}_g^\chi \cong (\mathcal{F}_1^{*g})^\chi \cong (\mathcal{F}_1^\chi)^{*g}.$$

Since $\mathcal{R}_K^g(c)$ is equivariantly formal, Propositions 2.24 implies $H^0((\mathcal{F}_1^\chi)^{*g}) \cong H^0(\mathcal{F}_1^\chi)^{\otimes g}$. $\square$

Proposition 3.18. *If K is of type (i) or (ii), then $\mathcal{F}_1$ is monodromy sheaf, with vertex stalk $\wedge(\mathfrak{t}^*) \otimes A$ and holonomy $S_{\alpha(e)} \otimes id_A$ for all non-degenerate edges e, where $S_{\alpha(e)} \in Aut(\wedge(\mathfrak{t}^*))$ is induced by the Weyl reflection in the hyperplane $\alpha(e)^\perp \subset \mathfrak{t}$.*

Proof. If $K \cong \tilde{K} \cong SU(2) \times U(1)^{r-1}$, the representation variety splits as a product $\mathcal{R}^1_{\tilde{K}}(c) = \mathcal{R}^1_{SU(2)}(c_1) \times \mathcal{R}^1_{U(1)^{r-1}}(c_2)$ and the result follows from the $SU(2)$ case, Proposition 3.8.

Otherwise we have $\mathcal{F}_1 = Ind_{\tilde{T}_2}^{T_2}(\tilde{\mathcal{F}}_1)$ where $\tilde{\mathcal{F}}_1$ where is a monodromy sheaf as described above. Because the identification of the vertex stalks $\tilde{\mathcal{F}}_1(v) \cong \wedge(\mathfrak{t}^*) \otimes A$ were specifically chosen to be invariant under the $\tilde{T}_2$-action, it is clear that $\mathcal{F}_1$ is also a monodromy sheaf with the required monodromy. $\square$

Restriction to the vertex v defines the inclusion

$$i^* : H^0(\mathcal{F}_1^\chi) \hookrightarrow \mathcal{F}_1^\chi(v) \cong \wedge(\mathfrak{t}^*) \otimes S(\mathfrak{t}^*).$$

We now describe the image of i^*.

Let $\alpha_0 \in \mathfrak{t}^*$ be a root of K and choose a basis $\{\alpha_0, \beta_1, ..., \beta_{r-1}\}$ of $\mathfrak{t}^*$ where the β_i are orthogonal to α_0 (alternatively, the β_i are $+1$ eigenvectors for the reflection defined by α_0). If $m \in \wedge(\mathfrak{t}^*)$ is a monomial in this basis define $deg_{\alpha_0}(m) = 1$ if α_0 is a factor of m and $deg_{\alpha_0}(m) = 0$ if not.

Proposition 3.19. *The image of $i^* = i^*_{\alpha_0}$ in $\wedge(\mathfrak{t}^*) \otimes S(\mathfrak{t}^*) \cong \mathcal{F}_1^\chi(v)$ is generated as a $S(\mathfrak{t}^*)$-submodule by generators*

$$m \otimes 1 \qquad \textit{if } (-1)^{deg_{\alpha_0}(m)}\chi(exp(\pi i h_\alpha)) = 1$$

and

$$m \otimes \alpha_0 \qquad \textit{if } (-1)^{deg_{\alpha_0}(m)}\chi(exp(\pi i h_\alpha)) = -1$$

as m varies over a monomial basis.

Proof. The basis is chosen so that for a monomial $m \in \wedge(\mathfrak{t}^*)$,

$$S_{\alpha(e)}(m) = (-1)^{deg_{\alpha_0}(m)} m.$$

If K has type (i) or (ii), T_2 factors as $T_2' \times T_2''$, where T_2'' acts freely on $\Gamma_{\mathcal{R}^1_K(c)}$ and $T_2' \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $exp(\pi i h_{\alpha_0})$, which fixes the non-degenerate edges and transposes their incident vertices. Thus we may restrict attention to a single edge, and identify $i^*_{\alpha_0}$ with the intersection of the image of the matrix

$$\begin{pmatrix} 1 & \alpha_0 \\ S_{\alpha_0} & -\alpha_0 S_{\alpha_0} \end{pmatrix}$$

in $(\wedge(\mathfrak{t}^*) \otimes A)^{\oplus 2}$ with the diagonal if $\chi(exp(\pi i h_{\alpha_0})) = 1$ or the anti-diagonal if $\chi(exp(\pi i h_{\alpha_0})) = -1$, from which the result follows.

In the remaining case $K \cong SO(3) \times U(1)^{r-1}$, T_2 acts freely, so $\mathrm{im}(i^*_{\alpha_0})$ is independent of χ. In particular $\mathcal{F}_1^\chi \cong \mathcal{F}_1^{T_2}$. From the construction in the proof of Lemma 3.14, we see that $\mathcal{F}_1^{T_2} = (\tilde{F}_1 \coprod \tilde{F}_1)^G = \tilde{\mathcal{F}}_1^{\tilde{T}_2}$ which is described above. Since $exp(i\pi h_{\alpha_0}) = exp(\mathbb{1}) = 1$ in K, this proves the result. ∎

Corollary 3.20. *For regular c, the Poincaré series of $H_T^*(\mathcal{R}_K^g(c))$ are*

$$\frac{P_t^T(\mathcal{R}_K^g(c))}{P_t(BT)} = \begin{cases} 2(1+t^3)^g & \textit{if } K = SO(3) \\ 2(1+t)^g((1+t^3)^g + (t+t^2)^g) & \textit{if } K = U(2) \end{cases} \tag{34}$$

Proof. If $K = SO(3)$, then $\mathfrak{t}^* \cong \mathbb{C}^1$ has basis $\{\alpha_0\}$ and $T_2 \cong \mathbb{Z}/2\mathbb{Z}$ has two characters $\chi \in \hat{T}_2$. In both cases

$$H^0(\mathcal{F}_1^\chi) \subset \wedge(\mathfrak{t}^*) \otimes A$$

is a free A-module with generators $1 \otimes 1$ and $\alpha_0 \otimes \alpha_0$ of degree 0 and $1+2=3$ respectively. By Lemma 3.17 $H^0(\mathcal{F}_g^\chi) \cong H^0(\mathcal{F}_1^\chi)^{\otimes g}$ so

$$P_t^T(\mathcal{R}^g_{SO(3)}(c)) = \bigoplus_{\chi \in \hat{T}_2} P_t(H^0(\mathcal{F}_g^\chi)) = 2(1+t^3)^g P_t(BT)$$

as desired.

If $K = U(2)$, then $\mathfrak{t}^* \cong \mathbb{C}^2$ has basis $\{\alpha_0, \beta_1\}$ and $T_2 \cong (\mathbb{Z}/2\mathbb{Z})^2$ has two characters satisfying $\chi(exp(\pi i h_{\alpha_0})) = 1$ and two satisfying $\chi(exp(\pi i h_{\alpha_0})) = -1$. In the first case $H^0(\mathcal{F}_1^\chi) \subset \wedge(\mathfrak{t}^*) \otimes A$ is generated by free generators

$$1 \otimes 1, \beta_1 \otimes 1, \alpha_0 \otimes \alpha_0 \text{ and } \alpha_0 \wedge \beta_1 \otimes \alpha_0$$

of degrees $0, 1, 3, 4$ respectively. In the second case $H^0(\mathcal{F}_1^\chi)$ is generated by a free basis

$$1 \otimes \alpha_0, \beta_1 \otimes \alpha_0, \alpha_0 \otimes 1 \text{ and } \alpha_0 \wedge \beta_1 \otimes 1$$

of degrees $2, 3, 1$ and 2 respectively. Again $H^0(\mathcal{F}_g^\chi) \cong H^0(\mathcal{F}_1^\chi)^{\otimes g}$, so

$$\frac{P_t^T(\mathcal{R}^g_{SO(3)}(c))}{P_t(BT)} = \bigoplus_{\chi \in \hat{T}_2} \frac{P_t(H^0(\mathcal{F}_g^\chi))}{P_t(BT)} = 2(1+t+t^3+t^4)^g + 2(t+2t^2+t^3)^g$$

as desired. ☐

Remark 12. Although Corollary 3.20 describes only the Betti numbers, the cup product structure of $H_T^*(\mathcal{R}_K^g(c)))$ is also completely determined by the proof.

Lemma 3.21. *The localization map $i^* : H^0(\mathcal{F}_g^{T_2}) \to \mathcal{F}_g^{T_2}(v)$ fits into a commutative diagram*

$$\begin{tikzcd} H^0(\mathcal{F}_g^{T_2}) \arrow[r, "i^*"] \arrow[d, "\cong"] & \mathcal{F}_g^{T_2}(v) \arrow[d, "\cong"] \\ H_T^*(K^g) \arrow[r, "j^*"] & H_T^*(T^g) \end{tikzcd} \tag{35}$$

where T acts on K^g by conjugation, $j : T^g \hookrightarrow K^g$ is inclusion of the T-fixed points and the isomorphism $\mathcal{F}_g^{T_2}(v) \cong H_T^*(T^g) \cong \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$ is described in Remark 5.

Proof. The action of T on K^g by conjugation is equivariantly formal so $H_T^*(K^g) \cong H^0(\mathcal{F}_{K^g})$ is a free A-module. The projection map $\mathcal{R}_K^g(c)$ sending $(k_0, ..., k_g)$ to $(k_1, ..., k_g)$ is K-equivariant, so it induces a homomorphism $H_T^*(K^g) \to H^0(\mathcal{F}_g)$ by Proposition 2.10. The image of this map consists of T_2-invariant sections, so we gain homomorphism $\phi : H_T^*(K^g) \to H^0(\mathcal{F}_g)^{T_2} \cong H^0(\mathcal{F}_g^{T_2})$ which fits into the commutative diagram (35). Because K^g is equivariantly formal, j^* is injective so ϕ must be injective. To see that ϕ is a isomorphism, it is enough to confirm that $H_T^*(K^g)$ and $H^0(\mathcal{F}_g^{T_2})$ have the same Poincaré series.

The Poincaré series of $H_T^*(K^g) = H^*(K^g) \otimes H^*(BT)$ is

$$\big((1+t^3)(1+t)^{r-1}\big)^g P_t(BT)$$

and this agrees with $H^0(\mathcal{F}_g^{T_2})$ when $K = SU(2)$ by [Bai10] §5.1 and $K = SO(3)$ or $U(2)$ by Corollary 3.20. The case of general K of semi-simple rank one follows easily. ☐

3.3.2 c is not regular

Let $c \in T$ be a not necessarily regular element with centralizer $Z(c) \subseteq K$ and define Weyl group at c by the formula $W_c := N_{Z(c)}(T)/T$. Because K has semi-simple rank one, $W_c = W = \mathbb{Z}/2\mathbb{Z}$ if c is not regular and W_c is trivial if c is regular. The T-action on $\mathcal{R}_K^g(c)$ extends to a $Z(c)$-action and there is a well known formula:

$$H^*_{Z(c)}(\mathcal{R}_K^g(c)) \cong H_T^*(\mathcal{R}_K^g(c))^{W_c}.$$

Proposition 3.22. *Let $\mathcal{R} := \mathcal{R}_K^g(c)$ for c not necessarily regular and let $\mathcal{F}_g$ be as defined in §3.3.1. Then W_c acts on both $\mathcal{F}_{\mathcal{R}}$ and $\mathcal{F}_g$ giving rise to an isomorphism of W_c-invariants,*

$$H_T^*(\mathcal{R})^{W_c} \cong H^0(\mathcal{F}_{\mathcal{R}})^{W_c} \cong H^0(\mathcal{F}_g)^{W_c}.$$

Proof. If W_c is trivial then c is regular and the statement is vacuous. So assume c is not regular so $W_c = W \cong \mathbb{Z}/2\mathbb{Z}$.

The twisted W-actions on $H_T^*(\mathcal{R})$ and on $\mathcal{F}_{\mathcal{R}}$ are the standard ones described in Proposition 2.28 determining the isomorphism

$$H_T^*(\mathcal{R})^W \cong H^0(\mathcal{F}_{\mathcal{R}})^W$$

It remains to construct the twisted W_c-action on $\mathcal{F}_g$ and prove the remaining isomorphism.

Consider first the case of $K = \tilde{K} \cong SU(2) \times U(1)^{r-1}$. In this case the representation variety splits as $\mathcal{R} \cong \mathcal{R}^g_{SU(2)}(c_1) \times \mathcal{R}^g_{U(1)^{r-1}}(c_2)$. If $\gamma : [0,1] \to T$ is a path connecting c to a regular value, then Lemma 3.4 and Proposition 3.10 imply that $\mathcal{F}_g$ acquires a twisted W-action and $H^0(\mathcal{F}_g)^W \cong H_T^*(\mathcal{R})^W$.

Now consider K of type (ii) or (iii), with double cover $\phi : \tilde{K} \to K$ fitting into the exact sequence (31). Denote $\mathcal{R} = \coprod_{\phi(\tilde{c})=c} \mathcal{R}^g_{\tilde{K}}(\tilde{c})$ which by (32) forms a C_2^{g+1}-Galois covering $\tilde{\mathcal{R}} \to \mathcal{R}$.

The elements $\tilde{c} \in \phi^{-1}(c)$ are either both regular or both non-regular. If they are both regular, then $\mathcal{R}$ is isomorphic as a T-space to $\mathcal{R}^g_K(c')$ for regular $c' \in K$ and consequently $\mathcal{F}_{\mathcal{R}} \cong \mathcal{F}_g$ even before taking W-invariants. Otherwise, both $\tilde{c} \in \phi^{-1}(c)$ are non-regular and W acts on both cofactors, so $H_T^*(\tilde{\mathcal{R}})^W \cong H^0(\tilde{\mathcal{F}}_g)^W \oplus H^0(\tilde{\mathcal{F}}_g)^W$ since we have already confirmed this isomorphism for $K = \tilde{K}$.

By Remark 9 the actions by C_2^{g+1} and W commute, on both $H_T^*(\mathcal{R})$ and on $\mathcal{F}_g$. Thus

$$\begin{aligned} H_T^*(\mathcal{R})^W \cong (H_T^*(\tilde{\mathcal{R}})^{C_2^{g+1}})^W \cong (H_T^*(\tilde{\mathcal{R}})^W)^{C_2^{g+1}} \cong (H^0(\tilde{\mathcal{F}}_g)^W \oplus H^0(\tilde{\mathcal{F}}_g)^W)^{C_2^{g+1}} \\ \cong ((H^0(\tilde{\mathcal{F}}_g) \oplus H^0(\tilde{\mathcal{F}}_g))^{C_2^{g+1}})^W \cong H^0(\mathcal{F}_g)^W \end{aligned}$$

□

Remark 13. It is useful to describe more explicitly the twisted W_c-action on $\mathcal{F}_g$. This action is completely determined by its restriction to the vertex set $\mathcal{V}$. We may identify the vertex set $\mathcal{V} \cong \{t \in T | t^2 \in c\}$ and W_c acts on $\mathcal{V}$ by restricting the standard action on T. The stalks over every $v \in \mathcal{V}$ are identified with $\mathcal{F}_g(v) = \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$ (see Corollary 3.2) on which W_c-acts by through $S_{\alpha_0} \otimes S_{\alpha_0}$, where S_{α_0} denotes the extension of the Weyl reflection on $\mathfrak{t}^*$ to algebra automorphisms of $\wedge(\mathfrak{t}^*)$ and $S(\mathfrak{t}^*)$.

For completeness, we list equivariant Poincaré series for $SO(3)$ and $U(2)$ representation varieties.

Corollary 3.23. *For c non-regular, the equivariant Poincaré series of $\mathcal{R}^g_K(c)$ satisfies*

$$\frac{P_t^{Z(c)}(\mathcal{R}_K^g(c))}{P_t(BZ(c))} = \begin{cases} 2(1+t^3)^g & \textit{if } K = SO(3) \textit{ and } c = \mathbb{1} \\ (1+t^3)^g(1+t^2) & \textit{if } K = SO(3) \textit{ and } c \neq \mathbb{1} \\ (1+t)^g\big(2(1+t^3)^g + (t+t^2)^g(1+t^2)\big) & \textit{if } K = U(2) \end{cases} \tag{36}$$

Proof. We use the formula

$$H^*_{Z(c)}(\mathcal{R}_K^g(c)) \cong H^0(\mathcal{F}_{\mathcal{R}_K^g(c)})^W \cong H^0(\mathcal{F}_1^{*g})^W.$$

established in Propositions 3.22 and 3.16.

If $K = SO(3)$ and $c \neq \mathbb{1}$, then W transposes the two connected components of the GKM-graph. Therefore

$$P_t(H^0(\mathcal{F}_1^{*g})^W) = 1/2(P_t(\mathcal{F}_1^{*g})) = \frac{(1+t^3)^g}{(1-t^2)} = \frac{(1+t^3)^g(1+t^2)}{(1-t^4)} = (1+t^3)^g(1+t^2)P_t(BZ(c))$$

as desired.

If $K = SO(3)$ and $c = \mathbb{1}$, then W acts trivially on $\hat{T}_2$ and on the vertices of the GKM-graph. Thus W preserves the summands

$$H^0(\mathcal{F}_1^{*g}) = \bigoplus_{\chi \in \hat{T}_2} H^0(\mathcal{F}_1^{\chi})^{\otimes g}$$

and acts by the restriction of the standard twisted action on $\wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$. The generators of $H^0(\mathcal{F}_1^{\chi})^{\otimes g} \subset \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$ are invariant under this action and the desired result follows.

If $K = U(2)$ then $\epsilon := exp(\pi i h_{\alpha_0}) = -\mathbb{1}$. The action of $W = \mathbb{Z}/2\mathbb{Z}$ on $\hat{T}_2$ interchanges the two characters satisfying $\chi(\epsilon) = -1$ so they contribute

$$\frac{(1+t)^g(t+t^2)^g}{(1-t^2)^2} = \frac{(1+t)^g(t+t^2)^g(1+t^2)}{(1-t^2)(1-t^4)} = (1+t)^g(t+t^2)^g(1+t^2)P_t(BU(2))$$

to the Poincaré series. The characters satisfying $\chi(\epsilon) = 1$ are fixed by W so they each contribute a term of the form $(H^0(\mathcal{F}_1^{\chi})^{\otimes g})^W$. The W-action on vertices has a fixed point, so the twisted action on $H^0(\mathcal{F}_1^{\chi})$ is induced by the standard one on $\wedge(\mathfrak{t}^*) \otimes S(\mathfrak{t}^*)$, and this action leaves the generators of $H^0(\mathcal{F}_1^{\chi})$ invariant. Thus each summand $(H^0(\mathcal{F}_1^{\chi})^{\otimes g})^W$ contributes $(1+t)^g(1+t^3)^g P_t(BU(2))$ to the Poincaré series. Adding up these contributions gives the desired result.

□

Remark 14. For $K = U(2)$ and for $K = SO(3)$ and $c = \mathbb{1}$, the centralizer $Z(c) = K$, the formula (36) also provides the Poincaré polynomial of ordinary cohomology $H^*(\mathcal{R}_K^g(c))$. In the remaining case, $K = SO(3)$ and c is rotation by 180 degrees about an axis, $Z(c)$ is disconnected and (36) is not equal to the Poincaré polynomial of $H^*(\mathcal{R}_K^g(c))$.

3.4 General compact connected K

In this section, let K denote an arbitrary compact connected Lie group with maximal torus T. Denote by Δ the set of roots of $T \subset K$ and by Δ_+ be a choice of positive roots, which we think of as representing elements of $\mathbb{P}(\Lambda)$.

For $\alpha_0 \in \mathbb{P}(\Lambda)$, denote by K_{α_0} the centralizer of $\ker(\alpha_0)$ in K. It is easy to see that

$$\mathcal{R}^g_K(c)^{\ker(\alpha_0)} = \mathcal{R}^g_{K_{\alpha_0}}(c) \subseteq \mathcal{R}^g_K(c). \tag{37}$$

Lemma 3.24. *The centralizer $K_{\alpha_0} \subseteq K$ is strictly larger than T if and only if $\alpha_0 \in \Delta_+$.*

Proof. Centralizers of tori in K are connected (see [BtD85] IV Theorem 2.3) so K_{α_0} is connected for any $\alpha_0 \in \mathbb{P}(\Lambda)$. Thus K_{α_0} is strictly larger than T if and only the adjoint action of H on $\mathfrak{k}$ has fixed point set larger than $\mathfrak{t}$. Since the roots $\Delta \in \Lambda$ record the weights of the adjoint action, the result follows. $\square$

Lemma 3.24 and (37) imply that the one-skeleton of $\mathcal{R}^g_K(c)$ is the pair

$$\Big(\bigcup_{\alpha_0 \in \Delta_+} \mathcal{R}^g_{K_{\alpha_0}}(c),\ \mathcal{R}^g_T(c) \Big). \tag{38}$$

Consequently, the construction of the GKM-sheaf $\mathcal{F}_{\mathcal{R}^g_K(c)}$ reduces to the torus and the semi-simple rank one cases.

Theorem 3.25. *For any compact connected K, $c \in T$ and $g \geq 1$, the GKM-graph of $\Gamma_{\mathcal{R}^g_K(c)}$ admits a T_2-action, with vertex set*

$$\mathcal{V} \cong \{t \in T | t^2 = c\}$$

upon which T_2 acts freely and transitively by multiplication. If $\alpha_0 \in \Delta_+$ and $exp(\pi i h_{\alpha_0}) \neq \mathbb{1} \in T_2$, then $\mathcal{E}^{\alpha_0}$ partitions the vertices into pairs

$$\{v, exp(\pi i \alpha_0)v\}.$$

For all other $\alpha_0 \in \mathbb{P}(\Lambda)$, the edge partition is degenerate.

Proof. The description of the vertex set is from Corollary 3.2. Lemma 3.24 reduces the description of the edge set reduces to the semisimple rank one case, Proposition 3.13. $\Gamma_{\mathcal{R}^g_K(c)}$ is T_2-equivariant because $\Gamma_{\mathcal{R}^g_{K_{\alpha_0}}(c)}$ is for all $\alpha_0 \in \mathbb{P}(\Lambda)$. $\square$

Theorem 3.26. *Let K be compact and connected and let $c \in T$ be regular. For $g \geq 1$, denote $\mathcal{F}_g := \mathcal{F}_{\mathcal{R}^g_K(c)}$. Then $\mathcal{F}_g$ is a T_2-equivariant GKM-sheaf and there is an isomorphism of GKM-sheaves*

$$\mathcal{F}_g \cong \mathcal{F}_1 * ... * \mathcal{F}_1 \cong (\mathcal{F}_1)^{*g}.$$

Proof. Restricting to vertices, there is a natural isomorphism

$$\mathcal{F}_g|_{\mathcal{V}} \cong (\mathcal{F}_1)^{*g}|_{\mathcal{V}} \tag{39}$$

as defined in Corollary 3.2. Proposition 3.16 combined with (37) provides an isomorphism $\mathcal{F}_g|_{\mathcal{V} \cup \mathcal{E}^{\alpha_0}} \cong (\mathcal{F}_1)^{*g}|_{\mathcal{V} \cup \mathcal{E}^{\alpha_0}}$ respecting (39). Gluing together completes the result. $\square$

Theorem 3.27. *For $\alpha_0 \in \Delta_+$, let $\mathrm{im}(i^*_{\alpha_0})$ denote the $S(\mathfrak{t}^*)$-submodule of $\wedge(\mathfrak{t}^*) \otimes S(\mathfrak{t}^*)$ described in Proposition 3.19. Then*

$$H^0(\mathcal{F}_1^\chi) \cong \bigcap_{\alpha_0 \in \Delta_+} \mathrm{im}(i^*_{\alpha_0}).$$

Proof. Since $\mathcal{F}_1^\chi(\mathcal{V} \cup \mathcal{E}^{\alpha_0})$ is equal for K and K_{α_0}, this result follows from Lemma 2.7 and Proposition 3.19. □

Every connected compact Lie group K possesses a finite covering group $\tilde{K} \to K$ with $\pi_1(\tilde{K})$ is torsion-free. We can exploit this fact to simplify arguments, much as we did in the previous section.

Lemma 3.28. *If K is a compact, connected Lie group for which $\pi_1(K)$ is 2-torsion free, then for all roots $\alpha_0 \in \Delta_+$, the fundamental group $\pi_1(K_{\alpha_0})$ is also 2-torsion-free. This implies that the exponential of the coroot $exp(\pi i h_\alpha) \in K$ does not equal the identity.*

Proof. Consider the exact sequence of homotopy groups

$$\pi_2(K) \to \pi_2(K/K_{\alpha_0}) \to \pi_1(K_{\alpha_0}) \to \pi_1(K)$$

Here $\pi_2(K) = 0$ by Whitehead's theorem, and $\pi_2(K/K_{\alpha_0}) \cong H_2^*(K/K_{\alpha_0}; \mathbb{Z})$ is torsion free because K/K_{α_0} admits a Bruhat decomposition into even dimensional cells. Since $\pi_1(K)$ contains no 2-torsion, we deduce that $\pi_1(K_{\alpha_0})$ does not either.

Since K_{α_0} has semisimple rank one, it is isomorphic one of the groups in Lemma 3.11 and is not equal $U(1)^{(r-1)} \times SO(3)$, so it is easy to verify that $exp(\pi i \alpha_0) \neq \mathbb{1}$. □

Theorem 3.29. *Let $c \in K$ be a regular element and suppose that $\pi_1(K)$ is 2-torsion free. Then the GKM sheaf of $\mathcal{F}_1 := \mathcal{F}_{\mathcal{R}^1_K(c)}$ is isomorphic to the monodromy sheaf with fibre $\wedge(\mathfrak{t}^*) \otimes A$, with monodromy $\rho(e)$ equal to $S_{\alpha(e)} \otimes Id_A$, where $S_{\alpha(e)}$ is induced by the reflection in $\alpha(e)^\perp \subset \mathfrak{t}$.*

Proof. The restriction of $\mathcal{F}_1$ to $\mathcal{V} \cup \mathcal{E}^{alpha_0}$ is naturally isomorphic to $\mathcal{F}_{\mathcal{R}^1_{K_{\alpha_0}}(c)}$ so this follows directly from Proposition 3.18. □

3.4.1 Weyl invariants

Theorem 3.30. *Let $\mathcal{R} = \mathcal{R}^g_K(c)$ where c is not necessarily regular and let $\mathcal{F}_g$ be as defined in §3.3.1. There is an action of $W_c = N_{Z(c)}(T)/T$ on both $\mathcal{F}_R$ and $\mathcal{F}_g$ giving rise to an isomorphism of W_c-invariants,*

$$H^0(\mathcal{F}_\mathcal{R})^{W_c} \cong H^0(\mathcal{F}_g)^{W_c}.$$

Proof. The centralizer $Z(c)$ acts on $\mathcal{R}$ by conjugation, and this determines a twisted W_c-action on $\mathcal{F}_\mathcal{R}$ as described in Proposition 2.28.

The action of W_c on the restriction $\mathcal{F}_g|_\mathcal{V}$ is the same as explained in Remark 13. Namely W_c acts on $\mathcal{V} \cong \{t \in T | t^2 = c\}$ by the restriction of the standard action on T, and W_c acts on stalks $\mathcal{F}_g(v) \cong \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$ by the tensor product of standard representations

of W_c on $\wedge(\mathfrak{t}^*)$ and on $S(\mathfrak{t}^*)$. To see that this action extends to all of $\mathcal{F}_g \cong (\mathcal{F}_1)^{*g}$, observe that W_c-respects the local system of the monodromy sheaf $\mathcal{F}_1$ because for any $w \in W_c$ and $e \in \mathcal{E}$

$$(w \otimes w)(S_{\alpha(e)} \otimes Id_A)(w^{-1} \otimes w^{-1}) = S_{w\alpha(e)w^{-1}} \otimes Id_A = S_{\alpha(w \cdot e)} \otimes A$$

as an automorphism of $\wedge(\mathfrak{t}^*) \otimes Id_A$.

Now let $i^*_{\alpha_0} : \mathcal{F}_{\mathcal{R}}(\mathcal{V} \cup \mathcal{E}^{\alpha_0}) \to \mathcal{F}_{\mathcal{R}}(\mathcal{V})$ and $j^*_{\alpha_0} : \mathcal{F}_g(\mathcal{V} \cup \mathcal{E}^{\alpha_0}) \to \mathcal{F}_g(\mathcal{V})$ denote the restriction maps and identify

$$\mathcal{F}_g(\mathcal{V}) = \mathcal{F}_{\mathcal{R}}(\mathcal{V}) = \mathbb{C}\mathcal{V} \otimes \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*).$$

By Proposition 3.22 we know that $\mathrm{im}(i^*_{\alpha_0})^{S_{\alpha_0} \otimes S_{\alpha_0}} = \mathrm{im}(j^*_{\alpha_0})^{S_{\alpha_0} \otimes S_{\alpha_0}}$, so

$$\begin{aligned} H^0(\mathcal{F}_{\mathcal{R}})^{W_c} &\cong \Big(\bigcap_{\alpha_0 \in \Delta_+} \mathrm{im}(i^*_{\alpha_0}) \Big)^{W_c} = \Big(\bigcap_{\alpha_0 \in \Delta_+} (\mathrm{im}(i^*_{\alpha_0})^{S_{\alpha_0} \otimes S_{\alpha_0}} \Big)^{W_c} \\ &= \Big(\bigcap_{\alpha_0 \in \Delta_+} \mathrm{im}(j^*_{\alpha_0})^{S_{\alpha_0} \otimes S_{\alpha_0}} \Big)^{W_c} \cong H^0(\mathcal{F}_g)^{W_c} \end{aligned}$$

□

The W_c-action on $\mathcal{F}_g$ does not always commute with the T_2-action. Instead W_c acts by conjugation on T_2, restricting the standard action of W_c on T. Consequently, $H^0(\mathcal{F}_g)^{W_c}$ decomposes into a sum over the orbit space T_2/W_c,

$$H^0(\mathcal{F}_g)^{W_c} \cong \bigoplus_{[\chi] \in T_2/W_c} (H^0(\mathcal{F}_g)^{W_c})^{[\chi]} \tag{40}$$

Corollary 3.31. *Let $(W_c)_\chi$ be the stabilizer of $\chi \in T_2$. The summands of equation (40) satisfy*

$$(H^0(\mathcal{F}_g)^{W_c})^{[\chi]} \cong H^0((\mathcal{F}_1^{*g})^\chi)^{(W_c)_\chi}.$$

Proof. The W_c-action permutes the summands of $\bigoplus_{\chi \in \hat{T}_2} H^0(\mathcal{F}_g)^\chi$ according to its action on $\hat{T}_2$, so

$$(H^0(\mathcal{F}_g)^{W_c})^{[\chi]} \cong H^0(\mathcal{F}_g^\chi)^{(W_c)_\chi}$$

and the result then follows from the isomorphism $\mathcal{F}_g^\chi = (\mathcal{F}_1^{*g})^\chi$ of Theorem 3.26. □

A choice of base vertex v_* determines an isomorphism

$$\mathcal{F}_g^\chi(\mathcal{V}) \cong \mathcal{F}_g(v_*) \cong \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*) \tag{41}$$

and it will be important to describe the twisted $(W_c)_\chi$-action on $\mathcal{F}_g^\chi(\mathcal{V})$ in terms of this identification.

Lemma 3.32. *Given a base vertex v_*, let*

$$\tau : (W_c)_\chi \to Aut(\wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*))$$

denote the induced twisted action. Let τ^{st} denote the standard twisted W-action on $\wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$, induced by the standard Weyl group action on $\mathfrak{t}$. Then $\tau_w^{-1}\tau_w^{st} = \chi(t) \in \{\pm 1\}$ where $t \in T_2$ satisfies $tv_* = wv_*$. In particular, $\tau = \tau^{st}|_{(W_c)_\chi}$ if $(W_c)_\chi$ fixes the base vertex v_* or χ is trivial.

Proof. It was explained in the proof of Theorem 3.30 that W acts on $\mathcal{F}_g^\chi(\mathcal{V}) \cong \mathbb{C}\mathcal{V} \otimes \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$ by the tensor product of τ^{st} with the action on $\mathbb{C}\mathcal{V}$ induced by W-acting on $\mathcal{V}$. If v_* is fixed by $(W_c)_\chi$ then the identification (41) is undisturbed and $\tau = \tau^{st}$. Otherwise it is necessary to correct by multiplying by t which introduces the factor $\chi(t)$. $\square$

3.4.2 The trivial character

In [Bai08] Chapter 8, it was shown that the projection map $\mathcal{R}_K^g(c) \to K^g$ sending $(k_0, ..., k_g)$ to $(k_1, ..., k_g)$ is a "cohomological covering map". In particular, this means that the induced map on cohomology is an injection $H^*_{Z(c)}(K^g) \hookrightarrow H^*_{Z(c)}(\mathcal{R}_K^g(c))$. The analogue of this for $\mathcal{F}_{\mathcal{R}_K^g(c)}$ is the following.

Proposition 3.33. *For any $c \in T$ and $g \geq 1$, the T_2-invariant summand equals $H^0(\mathcal{F}_g^{T_2})^{W_c} \cong H_T^*(K^g)^{W_c}$.*

Proof. The T-fixed point set $(K^g)^T \cong T^g$ so

$$\mathcal{F}_g^{T_2}(\mathcal{V}) \cong \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*) \cong H_T^*((K^g)^T).$$

The action of W_c on $\mathcal{F}_g^{T_2}(\mathcal{V}) \cong \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$ is the standard one by Lemma 3.32. Consequently, it is enough to show $H^0(\mathcal{F}_g^{T_2}) \cong H_T^*(K^g)$.

The conjugation action of T on K^g is equivariantly formal so $H_T^*(K^g) \cong H^0(\mathcal{F}_{K^g})$. The one-skeleton of K is $(\bigcup_{\alpha_0 \in \Delta} K_{\alpha_0}, T)$ so the result follows directly from Lemma 3.21. $\square$

Remark 15. Using Theorem 3.27, we identify $H_T^*(K) \cong H^0(\mathcal{F}_1^{T_2})$ as a submodule of $\wedge(\mathfrak{t}^*) \otimes S(\mathfrak{t}^*)$. The equivariant fundamental class of K spans the one-dimensional subspace

$$\wedge^r(\mathfrak{t}^*) \otimes f$$

where $f := \prod_{\alpha_0 \in \Delta_+} \alpha_0$

4 Worked examples

We focus our attention on simple groups K for which $\pi_1(K)$ has finite and odd cardinality. Let $\Phi = Span_{\mathbb{Z}}\{h_\alpha | \alpha \in \Delta_+\}$ denote the coroot lattice of K in $\mathfrak{t}$. Lemma 3.28 implies that the map $exp(\pi i \ \ ) : \Phi \to T_2$ defines an isomorphism

$$\Phi/2\Phi \cong T_2 \tag{42}$$

so in this situation we may compute $H^0(\mathcal{F}_1^\chi)$ directly in terms of the (co)root system.

For a rank r abelian Lie algebra $\mathfrak{t}$, let $proj_r$ denote projection from $\wedge(\mathfrak{t}^*) \otimes S(\mathfrak{t}^*)$ onto $\wedge^r(\mathfrak{t}^*) \otimes S(\mathfrak{t}^*)$, which we identify with $S(\mathfrak{t}^*)$.

Lemma 4.1. *Let $d \in A = \mathbb{C}[\mathfrak{t}]$ be a homogeneous element and M be a free A-submodule of the A-algebra $\wedge(\mathfrak{t}^*) \otimes A$, with homogeneous free generators $\{x_1, ..., x_r\}$ satisfying*

$$proj_r(x_i x_{r-j}) = (\delta_{i,j} + \text{ higher order terms })d$$

where $\delta_{i,j}$ is the Kronecker delta. Then the product module MM satisfies $proj_r(MM) \subseteq (d)$ and M is maximal among submodules satisfying this property.

Proof. The matrix $\frac{1}{d}[x_i x_{r-j}]$ is unipotent and upper triangular, so row reduction produces a second basis $\{x_1', ..., x_r'\}$ of M which is dual to the x_i in the sense that $proj_r(x_i x_j') = \delta_{i,j} d$. Now suppose there exists $y \in \wedge\mathfrak{t}^* \otimes Sym(\mathfrak{t}^*) \setminus M$ for which $proj_r(My) \subset (d)$. Then by subtracting a linear combination of x_i', we may assume that $proj_r(My) = 0$. But this implies that $y = 0$ because M is of maximal rank and the $proj_r$ pairing is nondegenerate. □

We will use this Lemma in the following fashion.

Lemma 4.2. *Use Theorem 3.27 to identify $H^0(\mathcal{F}_1^\chi)$ with a submodule of $\wedge(\mathfrak{t}^*) \otimes A$. Then*

$$proj_r(H^0(\mathcal{F}_1^\chi) H^0(\mathcal{F}_1^\chi)) \subseteq (\prod_{\alpha \in \Delta_+} \alpha).$$

Proof. The fact that $H^0(\mathcal{F}_1)$ is a T_2-equivariant algebra implies that $H^0(\mathcal{F}_1^\chi) H^0(\mathcal{F}_1^\chi) \subseteq H^0(\mathcal{F}_1^{T_2}) \subseteq \wedge(\mathfrak{t}^*) \otimes A$. By Remark 15, we know

$$proj_r(H^0(\mathcal{F}_1^{T_2})) = H^0(\mathcal{F}_1^{T_2}) \cap (\wedge^r(\mathfrak{t}^*) \otimes A) = (\prod_{\alpha \in \Delta_+} \alpha).$$

□

4.1 A_2 type

In this section we let $K = SU(3)$ or $PSU(3)$. Then because $\pi_1(K)$ has odd order, (42) holds. Let $\{e_1, e_2, e_3\}$ denote the standard basis in $\mathbb{C}^3$ with the standard pairing. Identify $\mathfrak{t} = (e_1 + e_2 + e_3)^\perp \subset \mathbb{C}^3$. We choose positive roots $\Delta_+ = \{\alpha_{i,j} = e_i - e_j | i < j\}$ and use the pairing to identify $\alpha_{i,j} = h_{\alpha_{i,j}}$. The group $\Phi/2\Phi \cong T_2 \cong (\mathbb{Z}/2)^2$ has coset representatives consisting of 0 and the three positive roots Δ_+. There are three nontrivial characters χ_1, χ_2, χ_3 of $T_2 \cong \Phi/2\Phi$, defined by $\chi_k([\alpha_{i,j}]) = 1$ if and only if i, j, k are pairwise distinct. The Weyl group S_3 acts on the three nontrivial characters χ_1, χ_2, χ_3 in the standard fashion.

Proposition 4.3. *The summands $H^0(\mathcal{F}_1^{\chi_k})$ are pairwise isomorphic free modules for $k = 1, 2, 3$. Under the injection*

$$i^* : H^0(\mathcal{F}_1^{\chi_3}) \hookrightarrow \wedge\mathfrak{t}^* \otimes S(\mathfrak{t}^*)$$

described in Theorem 3.27 and using coordinates $x_1 = \alpha_{1,3}$ and $x_2 = \alpha_{2,3}$, the free basis is

$$1 \otimes x_1 x_2, \qquad x_1 \wedge x_2 \otimes (x_1 - x_2), \tag{43}$$

$$x_1 \otimes x_2 + x_2 \otimes x_1, \qquad x_1 \otimes x_2(x_1 - x_2) - x_2 \otimes x_1(x_1 - x_2). \tag{44}$$

Proof. That the $H^0(\mathcal{F}_1^{\chi_k})$ are pairwise isomorphic follows from the fact that the χ_k are permuted by W.

Let M denote the submodule of $\wedge(\mathfrak{t}^*) \otimes A$ generated by (43). It is straightforward linear algebra to check that M is contained in the intersection $\bigcap_{\alpha_0 \in \Delta_+} \mathrm{im}(i^*_{\alpha_0})$ of Theorem 3.27, so we conclude that $M \subseteq H^0(\mathcal{F}_1^{\chi})$.

To prove $M \supseteq H^0(\mathcal{F}_1^{\chi})$ observe that M satisfies the conditions of Lemma 4.1 with $d = \prod_{\alpha_0 \in \Delta_+} \alpha_0 = x_1 x_2 (x_1 + x_2)$. Combining this with Lemma 4.2 completes the proof. □

Corollary 4.4. *For $g \geq 1$, the A-module $H^0(\mathcal{F}_g)$ is free with Poincaré series*

$$P_t(H^0(\mathcal{F}_g)) = ((1 + t^3 + t^5 + t^8)^g + 3(t^3 + 2t^4 + t^5)^g)P_t(BT).$$

Proof. By Proposition 3.33 we know $H^0(\mathcal{F}_g^{T_2}) = H^*_T(K^g)$ and this explains the term $(1+t^3+t^5+t^8)^g P_t(BT)$. The remaining three isotypical components $H^0(\mathcal{F}_g^{\chi}) \cong H^0(\mathcal{F}_1^{\chi})^{\otimes g}$ are isomorphic and have Poincaré series $(t^3 + 2t^4 + t^5)^g P_t(BT)$ by Proposition 4.3. □

Corollary 4.5. *Let $\epsilon \in Z(K)$ and $\mathcal{R} = \mathcal{R}^g_K(\epsilon)$. The $H^0(\mathcal{F}_R)^W$ is a free A^W-module with Poincaré series*

$$((1 + t^3 + t^5 + t^8)^g + (1 + t^2 + t^4)(t^3 + 2t^4 + t^5)^g)P_t(BK)$$

Proof. Observe that every $\epsilon \in Z(K)$ has a square root in $Z(K)$, so the $W_\epsilon = W$ action on vertices has a fixed point. By Lemma 3.32 the induced W-action on $\wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$ is the standard one.

Because W permutes the three non-trivial characters $\{\chi_1, \chi_2, \chi_3\}$, $H^0(\mathcal{F}_\mathcal{R})^W$ decomposes into a sum of two A^W-modules as explained in Corollary 3.31. The trivial character contributes the first summand $H^*_T(K^g)^W = H^*_K(K^g)$ with Poincaré series

$$(1 + t^3 + t^5 + t^8)^g P_t(BK).$$

The remaining summand is

$$(\bigoplus_{i=1}^{3} H^0(\mathcal{F}_\mathcal{R}^{\chi_i}))^W \cong (H^0(\mathcal{F}_1^{\chi_3})^{\otimes g})^{W_{\chi_3}},$$

where $W_{\chi_3} =< (1,2) >$ is the stabilizer of χ_3. The free basis (43) of $H^0(\mathcal{F}_1^{\chi_3})$ is invariant under $(1,2)$ so the basis of $H^0(\mathcal{F}_1^{\chi_3})^{\otimes g}$ is also invariant and we deduce that

$$\begin{aligned} P_t((H^0(\mathcal{F}_1^{\chi_3})^{\otimes g})^{W_{\chi_3}}) = (t^3 + 2t^4 + t^5)^g P_t(A^{W_{\chi_3}}) &= \frac{(t^3 + 2t^4 + t^5)^g}{(1 - t^2)(1 - t^4)} \\ &= (t^3 + 2t^4 + t^5)^g (1 + t^2 + t^4) P_t(BK) \end{aligned}$$

□

Proof of Theorem 1.2. In [Bai09] the Betti numbers of $H^*_{U(3)}(\mathcal{R}^g_{U(3)}(\mathbb{1}))$ were computed and it was shown to be torsion-free as an $H^*(BU(3))$-module. It is easily deduced that $H^*_{SU(3)}(\mathcal{R}^g_{SU(3)}(\mathbb{1}))$ is torsion-free and that

$$P_t^{SU(3)}(\mathcal{R}^g_{SU(3)}(\mathbb{1})) = ((1 + t^3 + t^5 + t^8)^g + (1 + t^2 + t^4)(t^3 + 2t^4 + t^5)^g)P_t(BSU(3)).$$

By Theorem 2.6 this means that there is an injective morphism of graded algebras

$$H^*_{SU(3)}(\mathcal{R}^g_{SU(3)}(\mathbb{1})) \hookrightarrow H^0(\mathcal{F}_{\mathcal{R}^g_{SU(3)}(\mathbb{1})})$$

which by comparing Betti numbers must be an isomorphism, and $H^0(\mathcal{F}_{\mathcal{R}^g_{SU(3)}(\mathbb{1})})$ is free by Corollary 4.5. ∎

4.2 B_2

The root system of $Spin(5)$ has positive roots $\{2e_1, 2e_2, -e_1+e_2, e_1+e_2\} \subset \mathbb{C}^2$, with corresponding coroots $\{e_1, e_2, e_1+e_2, e_1-e_2\}$ under the standard pairing. Thus $T_2 \cong \Phi/2\Phi$ is generated by $[e_1]$ and $[e_2]$, and so $[e_1+e_2] = [e_1-e_2] \in \Phi/2\Phi$. The character group $\hat{T}_2$ has four elements defined in the basis $\{[e_1],[e_2]\}$ by matrices $[1,1], [-1,-1], [1,-1], [-1,1]$, and the Weyl group interchanges only $[1,-1]$ with $[-1,1]$.

Proposition 4.6. *For $K = Spin(5)$ and all $\chi \in \hat{T}_2$, the A-submodule $H^0(\mathcal{F}_1^\chi) \subset \wedge(\mathfrak{t}^*) \otimes A$ is free. The generators of $H^0(\mathcal{F}_1^{[-1,-1]})$ are*

$$\begin{array}{cc} 1 \otimes e_1e_2, & e_1 \wedge e_2 \otimes (e_1+e_2)(e_1-e_2), \\ e_1 \otimes e_2 + e_2 \otimes e_1 & e_1 \otimes e_2(e_1^2 - e_2^2) + e_2 \otimes e_1(e_2^2 - e_1^2). \end{array}$$

Changing basis to $x = e_1+e_2$ and $y = e_1 - e_2$, the generators of $H^0(\mathcal{F}_1^{[-1,1]})$ are

$$\begin{array}{cc} 1 \otimes xy(x+y), & x \wedge y \otimes (x-y), \\ x \otimes y(x+y) + y \otimes x(x+y), & x \otimes y(x-y) - y \otimes x(x-y). \end{array}$$

Proof. Analogous to Proposition 4.3. ∎

Corollary 4.7. *For $K = Spin(5)$ and $g \geq 1$, the ring $H^0(\mathcal{F}_g)$ is a free module over A with Poincaré polynomial*

$$P_t(H^0(\mathcal{F}_g)) = (P_t(K)^g + 2(t^4 + 2t^5 + t^6)^g + (t^3 + t^4 + t^6 + t^7)^g)/(1-t^2)^2.$$

Furthermore, for the augmentation map $A \to \mathbb{C}$, the extension of scalars $H^0(\mathcal{F}_g) \otimes_A \mathbb{C}$ is a dimension $10g$ Poincaré duality ring over $\mathbb{C}$.

Proof. The calculation of the Poincaré series is analogous to Corollary 4.4. The orientation class for the Poincaré duality pairing is

$$(\wedge^r(\mathfrak{t}^*) \otimes \prod_{\alpha_0 \in \Delta_+} \alpha_0) \in H^0(\mathcal{F}_1^{T_2})$$

and the pairing is just the extension of scalars of $proj_r$. The generators listed in Proposition 4.6 provide the nondegenerate pairs. ∎

Remark 16. If $\mathcal{R}^g_{Spin(5)}(c)$ is equivariantly formal then its ordinary cohomology ring is isomorphic to $H^0(\mathcal{F}_g)\otimes_A \mathbb{C}$. Since we know that $\mathcal{R}^g_{Spin(5)}(c)$ is an orientable manifold of dimension 10g, Corollary 4.7 is consistent with the conjecture with $\mathcal{R}^g_{Spin(5)}(c)$ being equivariantly formal.

In terms of the basis e_1, e_2, the Weyl group for B_2 is generated by the reflections:

$$s_1 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, s_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, s_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Corollary 4.8. *Let $\epsilon \in Spin(5) = K$ be the nontrivial central element and let $\mathcal{R} = \mathcal{R}^g_K(\epsilon^i)$ for $i = 0$ or 1. The ring of Weyl invariants $(H^0(\mathcal{F}_{\mathcal{R}}))^{W_{(\epsilon^i)}}$ is a free module over $A^W = H^*(BK)$ with Poincaré series*

$$(P_t(K)^g + (t^3+t^4+t^6+t^7)^g + t^2(1+t^4)(t^4+2t^5+t^6)^g)P_t(BK)$$

if $g+i$ is even and

$$(P_t(K)^g + t^4(t^3+t^4+t^6+t^7)^g + t^2(1+t^4)(t^4+2t^5+t^6)^g)P_t(BK)$$

if $g+i$ is odd.

Proof. The generators of $H^0(\mathcal{F}_1^{[-1,-1]})$ are not invariant under the standard $W_{\epsilon^i} = W$ action, but instead are weight vectors, of weight -1 for s_1 and s_2 and weight $+1$ for s_3. Taking tensor powers, it follows that for even g, the free A-generators of $H^0(\mathcal{F}_g^{[-1,-1]})$ equal the free $A^W = H^*(BK)$ generators of $H^0(\mathcal{F}_g^{[-1,-1]})^{W_{\mathbb{1}}}$, while for odd g the generators of $H^0(\mathcal{F}_g^{[-1,-1]})^{W_{\mathbb{1}}}$ are e_1e_2 times the generators of $H^0(\mathcal{F}_g^{[-1,-1]})$. On the other hand, one checks that under the twisted action of W_ϵ on $H^0(\mathcal{F}_1^{[-1,-1]})$ the generators are invariant, so for odd g the generators of $H^0(\mathcal{F}_g^{[-1,-1]})^{W_\epsilon}$ coincide with those of $H^0(\mathcal{F}_g^{[-1,-1]})$ while for even g we must shift by e_1e_2.

The remaining pair of character are permuted by the Weyl group, so we must consider $(H^0(\mathcal{F}_g^{[1,-1]})\oplus H^0(\mathcal{F}_g^{[-1,1]}))^{W_\epsilon} \cong H^0(\mathcal{F}_g^{[1,-1]})^{W_{[1,-1]}}$, which will be a free module over $A^{W_{[1,-1]}}$. Similar considerations now apply, but in this case generators must be shifted by a degree two element (either e_1 or e_2 depending on parity of $g+i$). ☐

4.3 G_2

G_2 has root system spanning the orthogonal complement of $e_1+e_2+e_3$ in $\mathbb{C}^3$, with simple roots $\alpha = e_1-e_2$ and $\beta = -2e_1+e_2+e_3$ with remaining positive roots $\alpha+\beta$, $2\alpha+\beta$, $3\alpha+\beta$ and $3\alpha+2\beta$. The Weyl group is isomorphic to the dihedral group D_6 and acts transitively on the nontrivial characters of $T_2 \cong \Phi/2\Phi$, with stabilizer isomorphic to $\mathbb{Z}_2\oplus\mathbb{Z}_2$. If we let χ denote the nontrivial character sending h_α and $h_{3\alpha+2\beta}$ to 1 then W_χ is generated by matrices

$$s_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, s_2 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the basis $x = \alpha+\beta$ and $y = 2\alpha+\beta$.

Proposition 4.9. *For $K = G_2$, the module $H^0(\mathcal{F}_1^\chi)$ is a free A-module generated by*

$$1 \otimes xy(2x-y)(2y-x), \qquad (x \otimes y - y \otimes x)(x^2-y^2),$$
$$x \wedge y \otimes (x^2-y^2), \qquad (x \otimes y + y \otimes x)(2x-y)(2y-x).$$

Proof. Analogous to Proposition 4.3. □

Corollary 4.10. *The graded ring $H^0(\mathcal{F}_g)$ is a free A-module with Poincaré series $(P_t(K)^g + 3(t^6+2t^7+t^8)^g)P_t(BT)$. Moreover the extension of scalars by the augmentation map $H^0(\mathcal{F}_g) \otimes_A \mathbb{C}$ is a Poincaré duality ring of dimension $14g$.*

Proof. Analogous to Corollary 4.7. □

The simply connected group G_2 has trivial centre.

Corollary 4.11. *The ring $H^0(\mathcal{F}_{\mathcal{R}^g_{G_2}(\mathbb{1})})^W \cong (H^0(\mathcal{F}_g))^W$ is a free A^W-module with Poincaré series*

$$(P_t(G_2)^g + (1+t^4+t^8)t^{6g}(1+t)^g((t^4+t^2)(t+1)^g + (t^2-1)(t-1)^g)/2)P_t(BG_2).$$

Proof. We have $P_t(H^0(\mathcal{F}_g)^W) = P_t(K^g)P_t(BK) + P_t((H^0(\mathcal{F}_1^\chi)^{\otimes g})^{W_\chi})$. The formula follows by computation from Proposition 4.9. □

4.4 A_3

Let $\{e_1, e_2, e_3, e_4\}$ denote the standard basis in $\mathbb{C}^4$ with the standard pairing. Identify $\mathfrak{t} = (e_1+e_2+e_3+e_4)^\perp \subset \mathbb{C}^3$. We choose positive roots $\Delta_+ = \{\alpha_{i,j} = e_i - e_j | i < j\}$ and use the pairing to identify $\alpha_{i,j} = h_{\alpha_{i,j}}$. The Weyl group $W = S_4$ acts on the group of characters $\hat{T}_2 = \Phi/2\Phi$ with two nontrivial orbits, distinguished by where they send $-\mathbb{1}$.

Proposition 4.12. *Let $\chi : \Omega/2\Omega \to \mathbb{Z}/2$ be the nontrivial character sending $[h_{\alpha_{1,2}}], [h_{\alpha_{1,3}}], [h_{\alpha_{2,3}}]$ to 1 (χ lies in orbit of size 4). Then $H^0(\mathcal{F}_1^\chi)$ is a free A-module. Choosing coordinates $x_i = \alpha_{i,4}$, $i = 1, 2, 3$, we have free basis*

$$1 \otimes x_1x_2x_3, \qquad (x_1 \wedge x_2 \wedge x_3) \otimes (x_1-x_2)(x_2-x_3)(x_3-x_1),$$
$$\sum_{c.p} x_1 \otimes x_2x_3, \qquad \sum_{c.p.} (x_1 \wedge x_2) \otimes (x_1-x_2)x_1x_2x_3,$$
$$\sum_{c.p.} x_1 \otimes x_1x_2x_3, \qquad \sum_{c.p.} (x_1 \wedge x_2) \otimes (x_1-x_2)x_2x_3,$$
$$\sum_{c.p.} x_1 \otimes (x_2x_3)^2, \qquad \sum_{c.p.} (x_1 \wedge x_2) \otimes (x_1-x_2)x_3.$$

where the sums are over cyclic permutations by the 3-cycle $(1, 2, 3)$.

Proof. Analogous to Proposition 4.3. □

On the other hand, the remaining Weyl orbit of characters gives rise to non-free modules:

Proposition 4.13. *Let $\chi : \Phi/2\Phi \to \mathbb{Z}/2$ be the non-trivial character satisfying $\chi(h_{\alpha_{1,2}}) = \chi(h_{\alpha_{3,4}}) = 1$. Then $H^0(\mathcal{F}_1^\chi)$ is not free over A. The Hilbert series of $H^0(\mathcal{F}_1^\chi)$ is*

$$P_t(H^0(\mathcal{F}_1^\chi)) \cong \frac{(2t^7 + 4t^8 + 2t^9 + t^{10} + t^{11} - t^{12} - t^{13})}{(1-t^2)^3}.$$

Proof. The calculation was done using MAGMA [BCP97]. Because the numerator of the Hilbert series is has negative coefficients, the module can not be free. □

This example disproves the general conjecture that $\mathcal{R}_K^g(c)$ is equivariantly formal for all c and K, because it fails when $K = SU(4)$ and c is regular. However, when $c = \mathbb{1}$ there is no contradiction.

Proposition 4.14. *Let $\mathcal{R} = \mathcal{R}_K^1(\mathbb{1})$. The ring of Weyl invariants $H^0(\mathcal{F}_\mathcal{R})^W$* is *a free A^W-module, with Poincaré series*

$$\begin{aligned} P_t(H^0(\mathcal{F}_\mathcal{R})^W)P_t(BK) = P_t(K) &+ (1+t^2+t^4+t^8)(t^5+2t^6+t^7+t^8+2t^9+t^{10}) + \\ &(1+t^4+t^8)(t^8+t^{15}) + t(2t^{14}+2t^{12}+3t^{10}+t^8+t^6) + \\ &t^2(t^{20}+t^{16}+t^{14}+2t^{12}+2t^{10}+t^8+t^6) \end{aligned}$$

Proof. The first two terms are obtained in similar fashion to the cases A_2, B_2, C_2. The remaining terms were computed using MAGMA. □

5 Collecting data

The following tables list the Hilbert series of $H^0(\mathcal{F}_1)$ and whether or not it is a free A-module, for several values of simply connected K. All results not already described in §4 were obtained using MAGMA [BCP97].

The terms are collected into Weyl orbits of T_2-isotypical components. Every term possessing no negative coefficients corresponds to a free A-module summand.

Lie type	Free?	Hilbert series of $H^0(\mathcal{F}_{\mathcal{R}^1_K(c)})$ times $(1-t)^{\mathrm{rk}(K)}$ for regular c.
A_2	yes	$P_t(A_2)+3(t^3+2t^4+t^5)$
B_2	yes	$P_t(B_2)+2(t^4+2t^5+t^6)+(t^3+t^4+t^6+t^7)$
G_2	yes	$P_t(G_2)+3(t^6+2t^7+t^8)$
A_3	no	$P_t(A_3)+4(1+t)^2(t^8+t^5)+3(1+t)(-t^{12}+t^{10}+2t^8+2t^7)$
B_3	no	$P_t(B_3)+3t^7(t+1)(1+2t^3+t^6)+4t^9(t+1)^2(t^3+1)(-t^3+t^2+1)$
C_3	no	$P_t(C_3)+3t^8(t+1)^2(t^3+1)+3t^9(t+1)(-t^9+2t^5+t^2+t+1)+t^5(t+1)(t^3+1)(t^7+1)$
A_4	no	$P_t(A_4)+5t^7(t+1)^2(t^3+1)(t^5+1)+10t^{11}(t+1)^2(t^3+1)(-t^4+t+2)$
B_4	no	$P_t(B_4)+4t^{11}(t+1)(t^3+1)^2(t^7+1)+3t^{15}(t+1)(t^3+1)(-t^{11}+t^7+2t^3+t^2+1)+8t^{16}(t+1)^2(t^3+1)(-t^8+t^5+t^2+1)$
C_4	no	$P_t(C_4)+4t^{12}(t+1)^2(t^3+1)(t^7+1)+t^7(t+1)(t^3+1)(t^7+1)(t^{11}+1)+6t^{16}(t+1)^2(-t^{13}+3t^5+t^2+1)+4t^{15}(t+1)^2(t^3+1)^2(-t^9+t^8-t^7+2t^6-t^5+t^4-2t^3+t^2+1)$
D_4	no	$P_t(D_4)+12t^{11}(t+1)^2(t^3+1)(-t^8+t^7+t^2+1)+3t^{12}(t+1)^2(-3t^9+6t^5+t^2-t+1)$
F_4	no	$P_t(F_4)+3t^{15}(t+1)(t^3+1)(t^7+1)(t^{11}+1)+11t^{24}(t+1)^2(t^3+1)(-t^{14}+t^{11}-t^8+t^6+t^5-t^3+t^2+1)$

In the following table we collect Hilbert series for $H^0(\mathcal{F}_{\mathcal{R}^1_K(\mathbb{1})})^W$ divided by $P_t(BK)$, where $\mathbb{1}\in K$ is the identity.

Lie type	Free?	Hilbert series for $H^0(\mathcal{F}_{\mathcal{R}^1_K(\mathbb{1})})$ divided by $P_t(BK)$
A_2	yes	$P_t(A_2) + (t^4 + t^2 + 1)(t^5 + 2t^4 + t^3)$
B_2	yes	$P_t(B_2) + t^4(t^7 + t^6 + t^4 + t^3) + t^2(t^4 + 1)(t^6 + 2t^5 + t^4)$
G_2	yes	$P_t(G_2) + \frac{1}{2}t^6(t + 1)(t^8 + t^4 + 1)(t^5 + t^4 + 2t^3 - t + 1)$
A_3	yes	$P_t(A_3) + t^5(t + 1)^2(t^3 + 1)(3t^6 - t^5 + 2t^4 - t^3 + 3t^2 + 1)$
B_3	yes	$P_t(B_3)+t^7(t+1)^2(t^3+1)(t^{12}+2t^{10}-3t^9+3t^8+3t^6-3t^5+2t^4+2t^2-t+1)$
C_3	no	$P_t(C_3) + t^{10}(t + 1)(t^3 + 1)(-t^{17} + t^{14} + t^{13} - t^{11} + 3t^9 + 4t^8 - t^6 + 2t^5 + 2t^4 + t^3 + 2t + 1)$
A_4	no	$P_t(A_4)+(t+1)(t^3+1)(t^5+1)(t^4+t^3+t^2+t+1)(-t^{12}+t^9+t^8+2t^5+2t^4+1)$

Observe that this disproves the weak version of the conjecture, that $\mathcal{R}^g_K(\mathbb{1}) = Hom(\pi_1(\Sigma_g), K)$ is equivariantly formal for all K, because it fails when $g = 1$ and $K = SU(5)$.

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